Dynamical Systems 8-24-05

Dynamical systems are those which evolve in time. Here is a brief history.

- Mid 1600’s- Newton discovered his laws of motion
- Late 1800’s- Poincaré noticed that behavior of a system can depend heavily on the initial conditions. He introduced the idea of chaos for when a deterministic system depends heavily on initial conditions.
- First half of the 1900’s- Applications of nonlinear oscillators such as radio, laser, and radar emerge.
- 1963- Lorentz discovered chaotic motion/strange attraction from his study of convective rules in the atmosphere. This helps explain the unpredictability of the weather. A signature of chaos is recognized as that the solutions never become steady state or periodic state, but they oscillate in an irregular and aperiodic fashion. Also initial conditions that are “close” can lead to totally different solutions.
- 1970’s- It was found that there was structure in chaos. Ruelle and Takens studied turbulence in fluids. Feigenbaum showed that completely different systems “go chaotic” in the same way. Mandelbrot introduced fractals around the same time.
- 1980-90’s- Chaos and fractals are everywhere.

Linear systems are well studied and understood. They are easy to solve. On the other hand, each nonlinear system is unique, and each one requires individual attention.

There are two main kinds of dynamical systems: (1) Differential equations. These are continuous in time. (2) Iterated maps of the type $x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-k})$. These are discrete in time.

Among differential equations there are two types, namely, ODE’s and PDE’s. An example of an ODE is the damped harmonic oscillator

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.$$  

An example of a PDE is the heat equation

$$\frac{\partial u}{\partial t} \propto \frac{\partial^2 u}{\partial x^2}.$$  

We can also study systems of ODE’s such as

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\
\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n)
\end{align*}$$
Here $\dot{x} = \frac{dx}{dt}$. We can also take higher order ODE’s and convert them into a system of first order ODE’s. For example consider the damped harmonic oscillator again. If we let $x_1 = x$ and $x_2 = \dot{x}$, then we obtain the following system

$$\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1
\end{cases}$$

Note that the system above is linear. There are also nonlinear systems, an example of which is the nonlinear pendulum. The equation of motion is given by

$$\ddot{x} + \frac{g}{L} \sin(x) = 0,$$

where $x$ is the angle the pendulum makes with the vertical, $g$ is the acceleration due to gravity, and $L$ is the length of the pendulum.

If we now introduce the change of variables $x_1 = x$ and $x_2 = \dot{x}$, we obtain the following system

$$\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{L} \sin x_1
\end{cases}$$

Note that this system is nonlinear. We can, however, linearize it by using the approximation $\sin x \approx x$ for $|x| \ll 1$. The system then becomes

$$\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{L} x_1
\end{cases}$$

If the pendulum has small energy, this is a good approximation.

Given a system of ODE’s, such as the pendulum above, one can gain information about the system by looking at its phase portrait, or its trajectory in phase space. In the system above, we have that $x_1(t)$ is the position of the pendulum, and $x_2(t)$ is the velocity of the pendulum. Thus $(x_1(t), x_2(t))$ describes the motion in time of a point in the $(x_1, x_2)$ plane. The point moves along a curve starting at $(x_1(0), x_2(0))$. The curve is called a trajectory in phase space.

The phase space is $n$-dimensional, where $n$ is the order of the system. Here $n$ is 2.
Given a differential equation of the form $\dot{\vec{x}} = \vec{F}(\vec{x})$, the vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is a point in $n$-dimensional phase space. This point is moving along a curve that defines a trajectory in phase space. Given an initial state $\vec{x}(0)$, we want to find the future states. Different initial states will generate different trajectories. The system $\dot{\vec{x}} = \vec{F}(\vec{x})$ is often called a flow in phase space, by the following reasoning. Imagine that phase space is filled with fluid. Then $\dot{\vec{x}} = \vec{F}(\vec{x})$ gives the velocity of the fluid at point $\vec{x}$.

We will start by considering 1-dimensional systems, which are first order ODE’s. In this case, we have

$$n = 1, \quad \dot{x} = f(x), \quad x = x(t)$$

where $f$ and $x$ are real valued. We also assume that $f$ is a smooth function. We will often interpret a differential equation as a vector field. Consider as an example

$$\dot{x} = \sin x$$

We can integrate this to produce a solution:

$$\frac{dx}{dt} = \sin x \Rightarrow dt = \frac{dx}{\sin x}$$

$$\Rightarrow \int dt = \int \frac{dx}{\sin x} = \int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

If we impose the initial condition $x(0) = x_0$, we find that

$$C = \ln|\csc x_0 + \cot x_0|$$

so that

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

If instead we consider this ODE as a flow in phase space, we can obtain information from the graphical/geometric interpretation. If $x$ is a point on the real axis, then $\dot{x}$ is its velocity. The equation $\dot{x} = \sin x$ then gives us a vector field on a line. We draw arrows to indicate the velocity vector. If $\dot{x} > 0$, we draw an arrow to the right. If $\dot{x} < 0$, we draw an arrow to the left. If $\dot{x} = 0$, then we have a fixed point. We can classify fixed points as either stable or unstable.
Other names for stable fixed points are attractors, sinks, and equilibria points. Other names for unstable fixed points are repellers and sources. In the example above, if \( x(0) = \pi/4 \), then the velocity increases to 1, then decreases to 0 and the point stops at \( x = \pi \). By looking at the arrows, we see that if there is some small disturbance that moves the point away from \( \pi \), then the point will return. Thus \( x = \pi \) is a stable fixed point. If we take the information gained from the phase portrait and translate it into a graph of \( x \) versus \( t \), we get the following:

Consider the following phase portrait

As usual, we think of \( f(x) \) as the velocity of a fluid flowing along the real axis. If \( f(x) > 0 \), the flow is to the right, and if \( f(x) < 0 \) the flow is to the left. The fixed points \( x^* \) are solutions to \( f(x^*) = 0 \). Stable fixed points are when \( f'(x^*) < 0 \), and unstable fixed points occur when \( f'(x^*) > 0 \). We may characterize the stability of a fixed point by the following idea: if a fixed point is stable, perturbations will die out, and if it is unstable, a perturbation will grow. Other names for fixed points are equilibrium solutions, steady-state solutions, or rest solutions. These names make sense, because we see that if \( x = x^* \) for some value \( t \), then \( \dot{x} = 0 \), so \( x(t) = x^* \) for all \( t \).

**Example:** Find the fixed points for \( \dot{x} = x^2 - 1 \) and classify them.

To find \( x^* \), we solve \( f(x^*) = 0 \).

\[ x^{*2} - 1 = 0 \Rightarrow x^* = \pm 1 \]

Flow will be to the right where \( x^2 - 1 > 0 \) and to the left where \( x^2 - 1 < 0 \).
Also, we see from the phase portrait that small disturbances at \( x^* = -1 \) will die out (decay), but large disturbances (\( x > 1 \)) will not. Thus we call the fixed point \( x^* = -1 \) locally stable, not globally stable. Note that in the case of the sinusoid above, all the fixed points were locally stable.

**Example**: Given \( \dot{x} = x - \cos x \), find the fixed points and classify them.

Considering the following graph,

we see that \( f(x^*) = 0 \) has only one solution, and that \( x^* \) is an unstable fixed point.

**Example**: Consider the following circuit:

Here \( Q(t) \) is the charge of the capacitor, \( Q(0) = 0 \) when the switch is thrown at \( t = 0 \), \( I \) is the current, and charge accumulates in the capacitor at a rate of \( \dot{Q} = I \). Thus our equation is

\[
V = RI + \frac{Q}{C} = R\dot{Q} + \frac{Q}{C} \Rightarrow \dot{Q} = \frac{V}{R} - \frac{Q}{RC} = f(Q)
\]

Again, we solve \( f(Q^*) = 0 \) to find a fixed point \( Q^* \). The following graphs illustrate the behavior of the system. Note that in this case we have a globally stable fixed point.
Next we will study some population models. Here $N(t)$ is the population size at time $t$, $r$ is the growth rate, and the model is

$$\dot{N} = rN.$$ 

This model predicts exponential growth. The solution is

$$N(t) = N_0 e^{rt}$$

where $N_0 = N(0)$ is the initial population size. This model is somewhat simplistic, though. A model that is much closer to reality (and therefore more complicated) is the Logistic Equation. Now we assume that the growth rate is not constant, but that it depends on the size of the population.

$$r = r(N) = r \left( 1 - \frac{N}{k} \right) \Rightarrow \dot{N} = rN \left( 1 - \frac{N}{k} \right).$$

It is reasonable to assume here that $N \geq 0$. The fixed points of the equation are $N^* = 0$, which is unstable and $N^* = k$, which is stable.

Any small population will grow exponentially, then growth will slow as it nears the carrying capacity of the environment. Thus a nonzero population is always approaching its carrying capacity.

Another question that we would like answered is the following: If an equilibrium solution is perturbed slightly, how will it behave? Obviously this depends on the type of fixed point, but it also depends on how $f(x)$ behaves near the fixed point. To see this, we linearize near the fixed point as follows. Suppose $\dot{x} = f(x)$ as usual, and that $f(x^*) = 0$. Then $\eta(t) = x(t) - x^*$ is a perturbation from the fixed point. We wish to derive a differential equation for $\eta$.

$$\dot{\eta} = \frac{d}{dt}\eta = \frac{d}{dt}(x(t) - x^*) = \dot{x} = f(x) = f(\eta + x^*).$$

If we now Taylor expand $f$ about $x^*$, then

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + \mathcal{O}(\eta^2).$$
Since \( f(x^*) = 0 \), this term disappears, and if \( f'(x^*) \neq 0 \), then we may drop the \( O(\eta^2) \) terms to obtain

\[
\dot{\eta} \approx \eta f'(x^*).
\]

This is a linear equation in \( \eta \). As seen before in our simplistic population model, the solution is

\[
\eta = \eta_0 e^{kt}, \quad k = f'(x^*).
\]

If \( f'(x^*) > 0 \), \( \eta \) grows exponentially, so \( x^* \) is unstable. If \( f'(x^*) < 0 \), \( \eta \) decays, so \( x^* \) is stable. If \( f'(x^*) \neq 0 \), then we cannot use this analysis. In this case, the nonlinear terms cannot be disregarded, and nonlinear analysis is required.

**Definition 1** Given the differential equation \( \dot{x} = f(x) \) with a fixed point \( x^* \), the quantity

\[
\frac{1}{|f'(x^*)|}
\]

is called the characteristic time scale for the problem.

**Example**: For the logistic equation, we had

\[
\dot{N} = rN \left(1 - \frac{N}{k}\right).
\]

Which means that

\[
f(N) = rN - \frac{r}{k}N^2 \Rightarrow f'(N) = r - \frac{2r}{k}N.
\]

From above, we know that the fixed points of the equation were \( N^* = 0 \) and \( N^* = k \). Evaluating \( f' \) at these points, we find that \( f'(0) = r \) and \( f'(k) = -r \). Thus the characteristic time scale is \( 1/|r| \) for each fixed point. Also, we see that we have exponential growth in perturbations at \( N^* = 0 \), so it is unstable, and we have exponential decay in perturbations at \( N^* = k \), so it is stable.

**Example**: This example is to show that when \( f'(x^*) = 0 \), all bets are off.

(a) Consider \( \dot{x} = -x^3 \). Then \( f'(x^*) = 0 \) at \( x^* = 0 \). This corresponds to a stable fixed point as in the graph below.

(b) Consider \( \dot{x} = x^3 \). Then \( f'(x^*) = 0 \) at \( x^* = 0 \), but this time we have an unstable fixed point.

(c) Consider \( \dot{x} = x^2 \). Then \( f'(x^*) = 0 \) at \( x^* = 0 \), but this fixed point is half-stable, meaning that it attracts from one side and repels from the other.
(d) Consider $\dot{x} = 0$. Then any point $x^*$ is a stable fixed point, and perturbations neither decay nor grow.

**Example**: For $\dot{x} = (x - 1)(x - 2)^2(x - 3)^3$, the fixed point at 1 is stable, the fixed point a 2 is half-stable, and the fixed point at 3 is unstable.

In a one-dimensional problem, the phase point cannot change directions, so there can be no oscillations. The phase point always moves monotonically. Thus we cannot have periodic solutions because

$$\dot{x}|_{x=x_0} = f(x_0)$$

is a single valued function. This means that the particle cannot have both $\dot{x} > 0$ and $\dot{x} < 0$ at any point.

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Here is a statement of the existance and uniqueness theorem for the problem

$$\dot{x} = f(x), \ x(0) = x_0 \quad (1)$$

**Theorem 1** Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval $R$ containing $x_0$. Then the IVP (1) has a unique solution $x(t)$ on some time interval $(-\tau, \tau)$ containing $t = 0$.

**Example**: Consider $\dot{x} = x^{1/3}$, $x(0) = 0$. In this problem, the solution is not unique. We have a fixed point at $x = 0$, so $x(t) = 0$ is a solution of the problem. But we also have that

$$t + C = \int \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3}.$$

Applying the initial condition $x(0) = 0$ gives us that $C = 0$, so

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}.$$

Notice here that

$$f'(0) = \left(x^{1/3}\right)' \bigg|_{x=0} = \frac{1}{3}x^{-2/3} \bigg|_{x=0} = \infty,$$

and recall that the characteristic time scale is given by $\frac{1}{|f'(x^*)|}$ and the growth rate is $f'(x^*)$ near a fixed point. In this case, we have growth rate infinity and characteristic time scale 0.
**Example**: Consider $\dot{x} = 1 + x^2, \ x(0) = 0$. This equation is separable, so we may solve it as

$$
t + C = \int \frac{dx}{1 + x^2} \Rightarrow \arctan x = t + C \Rightarrow x(t) = \tan(t + C).
$$

Applying the initial condition $x(0) = 0$ gives us that $C = 0$. Then a graph of the solution is the following:

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If a solution reaches infinity in finite time, we call that situation blow up.

Given a problem $\dot{x} = f(x)$, consider the image of the phase particle moving down a curve as such:

This curve is called a potential function, and the dip is called a potential well. We define the potential function by

$$
f(x) = -\frac{dV}{dx}.
$$

The negative sign is to emphasize that in an autonomous system, the particle should always be losing potential energy. To see this, note that

$$
\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = -f(x)\dot{x} = -\left(\frac{dV}{dx}\right)^2 \leq 0.
$$

We know that $f(x^*) = 0$ corresponds to a fixed point, but now we see that this situation also corresponds to maxima or minima of the potential function. Local minima of $V(x)$ correspond to stable fixed points, and local maxima correspond to unstable fixed points.

**Example**: Consider $\dot{x} = x - x^3, x(0) = 0$. To find $V(x)$, we solve

$$
-\frac{dV}{dx} = x - x^3 \Rightarrow -V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4.
$$

This corresponds to a bi-stable system, or a system with two stable equilibria. The point $x^* = 0$ is unstable and $x^* = \pm 1$ are stable.
Next, we will move on to bifurcations. The simplest of these is the saddle-node bifurcation. Here fixed points can be viewed as being created or destroyed as a parameter in the differential equation changes. Consider as an example the problem $\dot{x} = r + x^2$. As $r$ increases, the curve $f(x)$ moves up and the fixed points move closer together. Finally at $r = 0$ they collide to form a half-stable fixed point, and then for $r > 0$, there are no fixed points.

We see that $r_c = 0$ is the critical value where the bifurcation (splitting into branches) of the solution occurs. We can think of this process as a stack of vector fields on the line,

which gives rise to the following diagram

We then rotate the axes of this graph to obtain what we refer to as the bifurcation diagram. The rotation is done because it is logical to think of the location of the fixed points as depending on the parameter $r$.

The saddle-node bifurcation is also sometimes referred to as the “blue sky bifurcation” because as $r$ decreases, a fixed point appears out of nowhere. Also note that $x^* = \pm \sqrt{-r}$ gives us an analytical formula for the fixed points and a shape for the bifurcation diagram.

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It so happens that all saddle node bifurcations behave like the representative equations
\[
\dot{x} = r - x^2, \quad \dot{x} = r + x^2,
\]
as is illustrated in the following graph:

We see that if we zoom in closely enough, the graph is similar to a parabola. **Example**: For the equation \( \dot{x} = rx - e^{-x} \), find the value of the parameter \( r \) at the bifurcation point.

To accomplish this, we plot \( r - x \) and \( e^x \) on the same coordinate axes. The intersections are fixed points.

Now note that as we vary \( r \), we get graphs that fit the following situations:

To find the critical value \( r_c \), we note that the graphs intersect tangentially at the bifurcation point. This gives us two conditions, and therefore two equations,
\[
\frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x) \quad \Rightarrow \quad -e^{-x} = -1 \quad \Rightarrow \quad x = 0
\]
and at that point the curves intersect, so
\[
e^{(0)} = r - (0) \quad \Rightarrow \quad r_c = 1
\]
If we now use the Taylor expansion for \( e^{-x} \),
\[
e^x = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} + \cdots.
\]
we find that
\[ \dot{x} = r - x - e^{-x} = r - x - \left[ 1 - x + \frac{x^2}{2!} + \cdots \right] = r - 1 - \frac{x^2}{2} + \mathcal{O}(x^3). \]

If we drop the higher order terms, we see that indeed, this bifurcation is closely related to those that we claimed were the prototypes for all saddle-node bifurcations. Actually, we can use the previous analysis to discover the normal form for any bifurcation. We examine \( f(x) \) near the bifurcation point \( (x^*, r_c) \) using the Taylor expansion.

\[
\dot{x} = f(x, r) = f(x^*, r_c) + (x - x^*) \frac{\partial f}{\partial x}(x^*, r_c) + (r - r_c) \frac{\partial f}{\partial r}(x^*, r_c) + \frac{(x - x^*)^2}{2} \frac{\partial^2 f}{\partial x^2}(x^*, r_c) + \cdots.
\]

To find where the fixed points are for a given value of \( r \), we need to solve \( f(x, r) = 0 \). The implicit function theorem tells us that there is a unique solution \( x = \varphi(r) \) if \( \frac{\partial f}{\partial x}(x_o, r_0) \neq 0 \). This implies that \( x_0 \) is not a bifurcation point. Thus, we require that \( \frac{\partial f}{\partial x}(x^*, r) = 0 \) at a bifurcation point. The Taylor series then becomes

\[
a(r - r_c) + b(x - x^*)^2 + \cdots
\]

because \( f(x^*, r_c) = 0 \) and \( \frac{\partial f}{\partial x}(x^*, r_c) = 0 \). If we rescale this properly, it becomes

\[ \dot{x} = r \pm x^2. \]

Note that in our example above, we had

\[ \frac{\partial f}{\partial x}(x^*, r_c) = -1 + e^{-x} = 0. \]

The next type of bifurcation that we will study is the transcritical bifurcation. In this type, the fixed point exists for all values of the parameter \( r \), but it changes stability type. The normal form is given by \( \dot{x} = r x - x^2 = -(x - \frac{r}{2})^2 + \frac{x^2}{4} \). The roots are given by \( x^- = 0 \) and \( x^+ = r \).

**Example**: Consider \( \dot{x} = r \ln x + x - 1 = (x - 1) - (-r) \ln x \).
If we let \( u = x - 1 \), then the equation becomes
\[
\dot{u} = r \ln(1 + u) + u = r \left( u - \frac{1}{2} u^2 + \mathcal{O}(u^2) \right) = (r + 1)u - \frac{r u^2}{2} + \mathcal{O}(u^3).
\]
This tells us that \( r_c = -1 \), where this polynomial has a repeated root. One root of this polynomial will be at \( u = 0 \), which corresponds to \( x = 1 \), and the other will be at
\[
u = \frac{2(r + 1)}{r},
\]
which corresponds to
\[
x = \frac{2r + 2}{r} + 1 = 3 + \frac{2}{r}.
\]
This gives the following bifurcation diagram:

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Another type of bifurcation is the pitchfork bifurcation. This occurs often in problems that have symmetry. The normal form for the pitchfork bifurcation is given by
\[
\dot{x} = rx - x^3.
\]
Note that if we replace \( x \) with \( -x \) in this formula, the system is unchanged, meaning that the system is invariant under the transformation \( x \mapsto (-x) \). The following are graphs of \( \dot{x} \) versus \( x \) for differing values of \( r \).

Note that in the \( r < 0 \) case, \( x^* = 0 \) is a stable fixed point, and it is still stable in the \( r = 0 \) case, but the decay of perturbations to the fixed point has slowed. This is called critical slowing. In the \( r > 0 \) case, the origin becomes unstable and two stable fixed points appear at \( x^* = \pm \sqrt{r} \). The bifurcation diagram is the following:
To further analyze the situation we referred to as critical slowing down, we will look at the solution to the normal form when $r = 0$. This gives us the problem

$$\dot{x} = -x^3, \ x(0) - x_0,$$

which we solve as

$$\frac{dx}{dt} = -x^3 \Rightarrow dt = -\frac{dx}{x^3} \Rightarrow t + C = \frac{x^{-2}}{2} \Rightarrow \frac{1}{x^2} = 2t + C.$$

Applying the initial condition tells us that $C = \frac{1}{x_0^2}$. Thus

$$x^2 = \frac{1}{2t + x_0} \Rightarrow x(t) = \frac{x_0}{\sqrt{1 + 2tx_0^2}}.$$

Using the fact that $\sqrt{1 + y} \approx 1 + \frac{1}{2}y$, we then have

$$x(t) = \frac{x_0}{1 + tx_0^2}.$$

This function approaches 0 as $t \to 0$, but much slower than exponentially. Thus the name critical slowing.

The next type of bifurcation we will study is the subcritical pitchfork bifurcation. The normal form is $\dot{x} = rx + x^3$. Here are some graphs of $\dot{x}$ versus $x$ for different values of $r$.

This gives the following bifurcation diagram:

**Example**: Consider $\dot{x} = rx + x^3 - x^5$. Some graphs for different $r$ look like the following:

And these lead us to this bifurcation diagram
This bifurcation diagram gives an example of what is known as hysteresis. We can see this phenomenon by following these steps:

1. Start at $x^* = 0$, which is stable for $r < r_s < 0$.
2. Begin to increase $r$. Note that $x^* = 0$ remains stable up to $r = 0$.
3. At $r = 0$, the origin loses its stability so that any small perturbation brings the system to a completely different state. At this stage the phase point would jump to one of the stable fixed points on the large amplitude branches with any perturbation.
4. As $r$ increases, the system continues along a large amplitude branch.
5. If we now begin to decrease $r$, the state stays on the large amplitude branch until $r = r_s$, where it jumps back to the origin as the large amplitude branch becomes unstable.

This lack of reversability is what we call hysteresis. There are other names for these bifurcations. The subcritical pitchfork is often referred to as hard or dangerous, and the supercritical pitchfork is called soft or safe.

**Imperfect Bifurcation and Catastrophe**

Consider the equation

$$\dot{x} = h + rx - x^3. \quad (2)$$

If $h = 0$, then (2) is the normal form for a supercritical pitchfork bifurcation. This bifurcation occurs in problems that have a lot of symmetry. However, the presence of $h$, known as the imperfection parameter, breaks this symmetry if $h \neq 0$.

If we fix $r$ and plot $y = rx - x^3$ and $y = -h$ on the same axes, then we get the following graphs:

To find $h_c(r)$, the critical value of $h$ where the saddle node bifurcation occurs as $r$ varies, we see that at that value is a maximum for the curve $y = rx - x^3$. Calculating the derivative at this point, we see that

$$\frac{d}{dx}(rx - x^3) = r - 3x^2.$$
Setting this equal to zero and solving gives us that \( x_{max} = \sqrt{\frac{r}{3}} \). Then

\[
y(x_{max}) = rx - x^3 \bigg|_{x=\sqrt{\frac{r}{3}}} = \frac{2r}{3} \sqrt{\frac{r}{3}} \Rightarrow h_c(r) = \pm \frac{2}{3\sqrt{3}} r^{\frac{3}{2}}
\]

If we graph \( h_c(r) \), we get the following

Here co-dimension refers to the dimension of the space we are working in minus the dimension of the object. Then, for example, a point in \( \mathbb{R}^2 \) has co-dimension 2 and a line or a curve has co-dimension 1. To get a more complete picture of what is going on, we can also graph \( x^* \) versus \( r \) as usual, by fixing a value of \( h \).

Similarly, we can plot \( x^* \) versus \( h \) for differing fixed values of \( r \).

Putting this information all together, we can construct a three dimensional plot of the situation.

Note that again we have uncovered a case of hysteresis. This is a consequence of the fold in the graph.

Next, we will look at a model for insect outbreak. This model is due to Ludwig, Jones, and Kelling (1978), and it was made to model the population of spruce budworms. One thing in particular that makes this model interesting is that there are two different time scales involved. The insects evolve on a very
short time scale, whereas the forest evolves on a much slower time scale. This means that the variables that describe the forest can be considered as drifting parameters. We let \( N(t) \) be the population size of the budworms. We assume that the environment permits the budworms to grow at rate \( R \) and carrying capacity \( K \). Then our model is

\[
\dot{N} = RN \left( 1 - \frac{N}{K} \right) - p(N),
\]

where \( p(N) \) is the death rate due to predation (birds). We will give \( p(N) \) the formula

\[
\frac{BN^2}{A^2 + N^2},
\]

which has the graph

The model now stands as

\[
\dot{N} = RN \left( 1 - \frac{N}{K} \right) - \frac{BN^2}{A^2 + N^2}.
\]

We would like a formulation where the variables are dimensionless. Thus we make the change of variables

\[
x = \frac{N}{A} \Rightarrow N = Ax.
\]

Then

\[
\frac{d}{dt}(Ax) = R Ax \left( 1 - \frac{Ax}{K} \right) - \frac{BA^2 x^2}{A^2 + A^2 x^2}.
\]

If we divide through by \( B \), this becomes

\[
\frac{d}{dt} \left( \frac{A}{B} x \right) = RA \frac{A}{B} x \left( 1 - \frac{Ax}{K} \right) - \frac{x^2}{1 + x^2}.
\]

Next, we scale time as

\[
\tau = \frac{B}{A} t \Rightarrow d\tau = \frac{B}{A} dt.
\]

Then we also have that

\[
\frac{A dx}{B dt} = \frac{A}{B} \frac{dx}{d\tau} = \frac{dx}{d\tau}.
\]

Finally, we introduce a dimensionless growth rate \( r = \frac{RA}{B} \) and a dimensionless carrying capacity \( \kappa = \frac{K}{A} \). After including these changes, our equation now becomes

\[
\frac{dx}{d\tau} = rx \left( 1 - \frac{x}{\kappa} \right) - \frac{x^2}{1 + x^2}.
\]
In the budworm model, the parameters $r$ and $\kappa$ depend on the forest, meaning that they change very slowly. Recall that the dimensionless equation was given by
\[
\frac{dx}{d\tau} = rx \left( 1 - \frac{x}{\kappa} \right) - \frac{x^2}{1 + x^2} = x \left[ r \left( 1 - \frac{x}{\kappa} \right) - \frac{x^2}{1 + x^2} \right].
\]
From this we see that $x = 0$ is a fixed point, and that it is unstable because near $x = 0$, the predation term is small, and thus the equation is approximately the same as the logistic equation. The other fixed points are given by
\[
r \left( 1 - \frac{x}{\kappa} \right) = \frac{x^2}{1 + x^2}.
\]
If $\kappa$ is small, then for any $r$, there is only one fixed point.

If $\kappa$ is large, then we have this graph:

If we start with 3 fixed points, as $r$ decreases, we have a saddle-node bifurcation: $b$ and $c$ move closer and then anhilate each other. If instead we increase $r$, there is another saddle-node bifurcation as $a$ and $b$ move closer, then anhilate. We call $a$ the refuge level, $b$ the threshold level, and $c$ the outbreak level. Here we can observe another hysteresis effect, in that even if $\kappa$ and $r$ are returned to the previous levels (before outbreak), the population will not return to its refuge level, because outbreak is a stable equilibrium.

To see the bifurcation curves, we write $r(x)$ and $\kappa(x)$ in parametric form as
\[
r \left( 1 - \frac{x}{\kappa} \right) = \frac{x}{1 + x^2}.
\]
At a saddle-node bifurcation, the curves must intersect tangentially, so we also have that
\[
\frac{d}{dx} \left( r \left( 1 - \frac{x}{\kappa} \right) \right) = \frac{d}{dx} \left( \frac{x}{1 + x^2} \right).
\]
If we solve these equations, we find that

\[ r(x) = \frac{2x^3}{(1 + x^2)^2}, \quad \kappa = \frac{2x^3}{x^2 - 1}. \]

The following graphs show us that we once again have a cusp catastrophe, in that the phase point jumps between different branches of the graph.

Dynamical Systems 9-14-05

**Flows On The Circle**

A flow on the circle has the form \( \dot{\theta} = f(\theta) \), but it is different from a flow on the line in that it always admits oscillations. Consider for example the following system.

\[ \dot{\theta} = \sin \theta \]

As usual, we say that \( \theta^* \) is a fixed point if \( f(\theta^*) = 0 \). The phase diagrams on the line and the circle look like

Suppose that \( -\infty < \theta < \infty \) and that \( \dot{\theta} = \theta \). This flow cannot be represented as a flow on the circle because for each choice of \( \theta_0 \), the velocity \( \dot{\theta} \) would take on different values at \( \theta_0 \) and \( \theta_0 + 2\pi \), which are the same point on the circle. Thus the velocity would not be single valued. One might think that this could be fixed by restricting \( \theta \) form \( -\pi \) to \( \pi \), but then we have a jump in the velocity between \( -\pi \) and \( \pi \), and we require that the vector field be smooth.

For a first order system \( \dot{\theta} = f(\theta) \), we define the vector field on the circle to be a real valued \( 2\pi \) periodic function \( f \), i.e.

\[ f(\theta_0) = f(\theta_0 + 2\pi k) \]

for any \( k \in \mathbb{Z} \).

**Uniform Oscillator**

The uniform oscillator is defined by

\[ \dot{\theta} = \omega \]
where \( \omega \) is constant. The solution is \( \theta = \omega t + \theta_0 \), and this describes uniform motion around the circle. As always, \( \omega \) is the angular frequency (velocity), and the motion is periodic with period \( T = \frac{2\pi}{\omega} \).

**Example**: Two joggers are running around a circular track. The number of seconds needed for a particular jogger to make a lap are given by \( T_1 \) and \( T_2 \) respectively. Let us assume that \( T_2 > T_1 \). If the runners start together, how long does it take for the first jogger to lap the second? In this situation we have that

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1, \quad \omega_1 = \frac{2\pi}{T_1}, \\
\dot{\theta}_2 &= \omega_2, \quad \omega_2 = \frac{2\pi}{T_2}.
\end{align*}
\]

The first runner will lap the second when the angle between them reaches \( 2\pi \). We define \( \varphi = \theta_1 - \theta_2 \). To see how long it takes for \( \varphi \) to reach the value \( 2\pi \), we calculate that

\[\dot{\varphi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2,\]

and that \( \varphi \) increases by \( 2\pi \) after

\[T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \left(\frac{\omega_1 - \omega_2}{2\pi}\right)^{-1} = \left(\frac{\omega_1}{2\pi} - \frac{\omega_2}{2\pi}\right)\left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}.
\]

This phenomenon is called the beat phenomenon. Oscillations with different frequencies periodically go into and out of phase.

**Non-Uniform Oscillator**

The equation for a non-uniform oscillator is given by

\[\dot{\theta} = \omega - a \sin \theta.
\]

Here we assume that \( a \geq 0 \). The graph of the phase circle looks like

![Graph of phase circle](image)

We note from this picture that a bifurcation occurs as \( a \) changes. The term \( a \sin \theta \) introduces a non-uniformity in that the fastest flow is at \( \theta = \frac{\pi}{2} \) and the slowest flow is at \( \theta = \frac{\pi}{2} \).

**Example**: Consider the overdamped pendulum.
The equation of motion is

\[ mL^2 \ddot{\theta} + b \dot{\theta} + mgL \sin \theta = \Gamma, \]

where \( b \) is the viscous damping constant and \( \Gamma \) is a constant torque that is applied in the counter-clockwise direction. In the overdamped limit, i.e. for large \( b \), the inertial term is negligible, so this reduces to

\[ b \dot{\theta} + mgL \sin \theta = \Gamma. \]

To non-dimensionalize, we divide through by \( mgL \) to get

\[ \frac{b}{mgL} \dot{\theta} \frac{\Gamma}{mgL} - \sin \theta, \]

and we introduce scaled time

\[ \tau = \frac{mgL}{b} t \Rightarrow \frac{d\theta}{d\tau} = \gamma - \sin \theta, \]

where \( \frac{\Gamma}{mgL} = \gamma \). Then for differing values of \( \gamma \), we have the following phase diagrams

As \( \gamma \to 1^+ \), the pendulum takes longer and longer to pass the point \( \theta = \frac{\pi}{2} \).

Dynamical Systems 9-16-05

We are considering non-uniform flows on the circle. Recall that the equation of interest was \( \dot{\theta} = \omega - a \sin \theta \), and that the phase diagram for this equation looked like

If \( a < \omega \), we can find the period of oscillation \( T \) as

\[ T = \int_{0}^{2\pi} dt = \int_{0}^{2\pi} \frac{dt}{d\theta} d\theta = \int_{0}^{2\pi} \frac{1}{\omega - a \sin \theta} d\theta = \frac{2\pi}{\sqrt{\omega^2 - a^2}}. \]
The last integral was evaluated using \( u = \tan \frac{\theta}{2} \). As \( a \to \omega^- \), we see that \( T \) diverges to \( \infty \), and when \( a \) is near \( \omega \)

\[
\sqrt{\omega^2 - a^2} = \sqrt{\omega + a \sqrt{\omega - a}} \approx \sqrt{2 \omega \sqrt{\omega - a}}.
\]

It is almost as if the fixed point were still present, because the velocity \( \omega - a \) is small. The phase partice spends a lot of time in what is known as the bottleneck.

\[
T \approx \frac{2\pi}{\sqrt{2\omega \sqrt{\omega - a}}}.
\]

We see that \( T \) blows up like \((a_c - a)^{\frac{1}{2}}\), which is called the square root scaling law. Here a saddle-node bifurcation takes place and the fixed points dissapear, but the “ghost” remains.

Recalling that the normal form for a saddle-node bifurcation is \( x = r + x^2 \), we can approximent the amount of time that \( T \) spends in the bottleneck by

\[
T_{\text{bottleneck}} \approx \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}.
\]

Here again we see the square root scaling law. Of course we are also assuming that the particle spends most of its time in the bottleneck.

**Synchronization**

The model that we will study here to demonstrate synchronization was given by Ermentrout and Rinzel in 1984. Let \( \theta(t) \) be the phase of fireflies flashing. In the absence of any stimulus, they flash at \( \theta = 0 \) with frequency \( \omega \), so

\[
\dot{\theta} = \omega, \quad \theta(0) = 0.
\]

If we introduce a periodic stimulus via

\[
\dot{\theta} = \Omega, \quad \theta(0) = 0,
\]

then the equation for the fireflies becomes

\[
\dot{\varphi} = \dot{\theta} - \omega = (\Omega - \omega) - A \sin \varphi.
\]

Here \( A > 0 \) is the resetting strength, which characterizes the ability of the firefly to change the frequency of its flashing. This tells us that if \( 0 < \Theta - \theta < \pi \), then the fireflies try to speed up. If we define \( \varphi = \Theta - \theta \), then the equation for \( \varphi \) is

\[
\dot{\varphi} = \dot{\Theta} - \dot{\theta} = (\Omega - \omega) - A \sin \varphi.
\]
We recognize this as a non-uniform oscillator. Next, we non-dimensionalize it. Making the changes $\tau = At$ and $\mu = \frac{\Omega - \omega}{A}$, we find that

$$\varphi' = \frac{d\varphi}{d\tau} = (\Omega - \omega - A \sin \varphi) \frac{1}{A} = \frac{\Omega - \omega}{A} - \sin \varphi = \mu - \sin \varphi.$$  

Here $'$ denotes differentiation with respect to $\tau$. Some graphs of the phase line for differing values of $\mu$ are the following.

Note that in the $\mu = 0$ case, the origin is a stable fixed point, and all nearby trajectories flow to the origin (or to another stable fixed point at an even multiple of $2\pi$), which means that the fireflies all start to flash in unison with the stimulus. This is called entraining with the stimulus. In the case $0 < \mu < 1$, the positive stable fixed point means that there is a constant difference between the stimulus and the fireflies. We say that the fireflies are phase-locked to the stimulus. Also note that at $\mu = 1$, we have a saddle-node bifurcation, and if $\mu > 1$, then the phase locking is lost, and there are no fixed points.

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Dynamical Systems 9-19-05

2D Flows: Linear Systems

In one dimension, we had equations of the form $\dot{x} = f(x)$. In two dimensions, we will have equations whose general form is

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}.$$  

In the case of a linear system, the equations will have the form

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}.$$  

We can write such a system in matrix form as

$$\dot{x} = Ax,$$  

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where

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \]

If a system is linear, and if \( \vec{x}_1 \) and \( \vec{x}_2 \) are solutions, then so is any vector of the form \( c_1 \vec{x}_1 + c_2 \vec{x}_2 \) for constants \( c_1 \) and \( c_2 \). Also, the origin is always a fixed point of any linear system because \( A\vec{0} = \vec{0} \).

**Harmonic Oscillator**

The vibration of a mass \( m \) on a linear spring with spring constant \( k \) is described by the equation

\[ m\ddot{x} + kx = 0. \]

The state of the system at any given time is characterized by the position \( x \) and the velocity \( y \). If we change the above equation to a system, we obtain

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\frac{k}{m}x
\end{align*}
\]

where \( \omega^2 = \frac{k}{m} \). We can now assign a vector field to this equation via

\( (\dot{x}, \dot{v}) = (v, -\omega^2 x) \)

On the \( x \) axis, \( v = 0 \), so this becomes \( (\dot{x}, \dot{v}) = (0, -\omega^2 x) \), and on the \( v \) axis, \( x = 0 \), so we have \( (\dot{x}, \dot{v}) = (v, 0) \). This gives us information we can use to sketch the phase diagram.

Indeed, we can verify that the orbits are actually ellipses. We have that

\[ \frac{dv}{dx} = \frac{\dot{x}}{\dot{v}} = -\omega^2 x \]

Which implies that

\[
\int -\omega^2 x \, dx = \int v \, dv. \quad \Rightarrow \quad \frac{\omega^2 x^2}{2} + \frac{v^2}{2} = C.
\]
Example: Consider the problem \( \dot{x} = Ax \), where
\[
A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}.
\]
This corresponds to the system
\[
\begin{align*}
\dot{x} &= ax \\
\dot{y} &= -y.
\end{align*}
\]
We see that the system is decoupled, meaning that the time derivative of each variable depends only on itself. We take as initial conditions \( x(0) = x_0 \) and \( y(0) = y_0 \). By elementary means, we find that the corresponding solution is
\[
\begin{align*}
x(t) &= x_0 e^{at} \\
y(t) &= y_0 e^{-t}.
\end{align*}
\]
We also note that \( y \) decays exponentially as \( t \to \infty \). The behavior of \( x \) depends on \( a \). If \( a < 0 \), then \( x(t) \) also decays exponentially. In this case, all trajectories approach the origin as \( t \to \infty \). If, in particular, \( a < -1 \), then \( x(t) \) decays more rapidly than \( y(t) \). Trajectories approach the origin tangent to the slower direction.

If one looks backward along the trajectory, all trajectories become parallel to the fastest decaying trajectory (\( x \) direction). Here \( \vec{x} = 0 \) is a fixed point that is called a stable node.

If \( a = -1 \), then
\[
\frac{y(t)}{x(t)} = \frac{x_0}{y_0},
\]
which is constant along a trajectory. This gives us the following phase diagram

Here \( \vec{x}^* = 0 \) is called a stable symmetric node or "star".

If \( -1 < a < 0 \), then once again we have a stable node, but trajectories approach the origin along the \( x \) direction.
In the case $a = 0$, we have that $x(t) = x_0 e^{at} = x_0$, so we have a line of fixed points along the $x$ axis, and the trajectories approach the same points along vertical lines.

If $a > 0$, then $\vec{x}^* = 0$ is unstable. In forward time ($t \to \infty$), we have trajectories that are asymptotic to the $x$-axis. In backward time ($t \to -\infty$), trajectories are asymptotic to the $y$ axis. In this case $\vec{x}^* = 0$ is called a saddle.

**Definition 2** The stable manifold of a fixed point $\vec{x}^*$ is the set
\[
\{(x_0, y_0) : \vec{x}(t) \to \vec{x}^* \text{ as } t \to \infty\},
\]
and the unstable manifold of $\vec{x}^*$ is the set
\[
\{(x_0, y_0) : \vec{x}(t) \to \vec{x}^* \text{ as } t \to -\infty\}.
\]

In the case $a > 0$ above, the $x$ axis was the unstable manifold and the $y$ axis was the stable manifold of $\vec{x}^* = 0$.

**Definition 3** A fixed point $\vec{x}^*$ is an attractor or attracting fixed point if any trajectory that starts near $\vec{x}^*$ approaches $\vec{x}^*$ as $t \to \infty$. If $\vec{x}^*$ attracts all trajectories in the plane, then it is called globally attracting.

Note that in the cases $a < 0$, $\vec{x}^* = 0$ was an attracting fixed point.

**Classification of Linear Systems**

Given the system $\vec{x}' = A\vec{x}$, we seek a solution of the form
\[
\vec{x}(t) = e^{\lambda t} \vec{v},
\]
where $\vec{v} \neq 0$ and $\lambda$ is the growth rate. If such a $\lambda$ and $\vec{v}$ exist, then $\vec{x}'(t) = \lambda e^{\lambda t} \vec{v}$, and substituting this into the original equation, we find that
\[
\lambda e^{\lambda t} \vec{v} = Ae^{\lambda t} \vec{v} \Rightarrow A\vec{v} = \lambda \vec{v},
\]
so $\lambda$ is an eigenvalue of $A$ corresponding to eigenvector $\vec{v}$. To find $\lambda$ and $\vec{v}$, we form the characteristic equation $\det(A - \lambda I) = 0$. We have that
\[
\det(A - \lambda I) = \det \left( \begin{array}{cc} a - \lambda & b \\ c & d - \lambda \end{array} \right) = (a-\lambda)(d-\lambda) - bc = ad-\lambda(a+d)+\lambda^2-bc
\]
which gives us that
\[ \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \]
where \( \tau \) is the trace of \( A \) and \( \Delta \) is the determinant of \( A \). If \( \lambda_1 \neq \lambda_2 \), i.e. the eigenvalues are distinct, then \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent, and for any starting vector \( x_0 \), we can represent it as \( \vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 \) for some \( c_1 \) and \( c_2 \). This implies that the solution is \( \vec{x}(t) = c_1e^{\lambda_1t}\vec{v}_1 + c_2e^{\lambda_2t}\vec{v}_2 \).

Example: Consider the system
\[
\begin{align*}
\dot{x} &= x + y \\
\dot{y} &= 4x - 2y
\end{align*}
\]
with initial condition \( (x_0, y_0) = (2, -3) \).

Then the corresponding matrix system is
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
1 & 1 \\
4 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
\]

First, we must find the eigenvalues of the matrix in the system. We note that \( \tau = -1 \) and \( \Delta = -6 \), so the characteristic equation is \( 0 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0 \), so \( \lambda_1 = 2 \) and \( \lambda_2 = -3 \). Next, we find the eigenvectors. The equation we must solve is
\[
\begin{pmatrix}
-1 & 1 \\
4 & -4
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
which we see has solution \( v_1 = v_2 = 1 \). To find an eigenvector for \( \lambda_2 \), we must solve
\[
\begin{pmatrix}
4 & 1 \\
4 & 1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
which has solution \( v_1 = 1, v_2 = -4 \). The general solution is a combination of the eigensolutions,
\[
\vec{x}(t) = c_1e^{\lambda_1t}\vec{v}_1 + c_2e^{\lambda_2t}\vec{v}_2 = c_1e^{2t}\begin{pmatrix}
1 \\
1
\end{pmatrix} + c_2e^{-3t}\begin{pmatrix}
1 \\
-4
\end{pmatrix}.
\]

If we apply the initial condition, we find that at \( t = 0 \)
\[
\vec{x}(0) = c_1\begin{pmatrix}
1 \\
1
\end{pmatrix} + c_2\begin{pmatrix}
1 \\
-4
\end{pmatrix} = \begin{pmatrix}
2 \\
-3
\end{pmatrix}.
\]

This implies that
\[
\begin{align*}
c_1 + c_2 &= 2 \\
c_1 - 4c_2 &= -3
\end{align*} \quad \Rightarrow \quad \begin{align*}
c_1 &= 1 \\
c_2 &= 1
\end{align*}.
\]
Thus

\[ \vec{x}(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} + e^{-3t} \\ e^{2t} - 4e^{-3t} \end{pmatrix}. \]

In the phase portrait, the solution corresponding to \( \lambda_1 = 2 \) is growing exponentially, and the solution corresponding to \( \lambda_2 = -3 \) is decaying exponentially. The origin is called a saddle in this situation.

The stable manifold of a fixed point \( \vec{x}^* \) is the set if initial conditions corresponding to trajectories such that \( \vec{x}(t) \to \vec{x}^* \) as \( t \to \infty \). In this case, the stable manifold is spanned by \( \vec{v}_2 \), the exponentially decaying eigensolution. The unstable manifold of a fixed point \( \vec{x}^* \) is the set of initial conditions corresponding to trajectories such that \( \vec{x}(t) \to \vec{x}^* \) as \( t \to -\infty \). In this case, the unstable manifold is the span of \( \vec{v}_1 \). As \( t \to \infty \), the trajectories approach the unstable manifold. As \( t \to -\infty \), the trajectories approach the stable manifold.

Suppose we have \( \lambda_1 < \lambda_2 < 0 \). Then the origin is a stable node, and the trajectories approach the origin tangent to the slowest direction (\( \vec{v}_1 \)). As \( t \to -\infty \), the trajectories are parallel to the eigenvector \( \vec{v}_2 \).

If \( 0 < \lambda_1 < \lambda_2 \), then we have an unstable node. The directions of the arrows are simply reversed from the phase diagram above.
Complex Eigenvalues

Recall that the characteristic polynomial of a linear system is \( \lambda^2 - \tau \lambda + \Delta = 0 \), which has solutions

\[
\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.
\]

If \( \tau^2 - 4\Delta < 0 \), then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are complex, say \( \lambda_{1,2} = \alpha \pm i\omega \) where \( \alpha = \frac{\tau}{2} \) and \( \omega = \frac{\sqrt{4\Delta - \tau^2}}{2} \). Since \( \omega \neq 0 \), the eigenvalues are distinct. The general solution is then given by

\[
\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.
\]

We will also have that \( \vec{v}_2 = \vec{v}_1 \), and we can require that \( c_2 = c_1 \) for real solutions. In this case, the solution is

\[
\vec{x}(t) = c_1 \vec{v}_1 e^{\alpha t} + c_2 \vec{v}_2 e^{\alpha t} = c_1 \vec{v}_1 e^{\alpha t} (\cos(\omega t) + i\sin(\omega t)) + c_2 \vec{v}_2 e^{\alpha t} (\cos(\omega t) - i\sin(\omega t)).
\]

If we let \( \vec{A} = c_1 \vec{v}_1 + c_1 \vec{v}_1 \) and \( \vec{B} = i(c_1 \vec{v}_1 + c_1 \vec{v}_1) \), both of which are real vectors, then this becomes

\[
e^{\alpha t} (\vec{A} \cos(\omega t) + \vec{B} \sin(\omega t))
\]

If \( \alpha < 0 \), then we have decaying oscillations. If \( \alpha = 0 \), then we have periodic solutions, and if \( \alpha > 0 \), then we have growing oscillations, as seen in the following phase diagrams:

If we have two equal eigenvalues \( \lambda_1 = \lambda_2 = \alpha = \frac{\tau}{2} \), then there are two cases. (1) There are two linearly independent eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \). In this case any initial vector \( \vec{x}_0 \) can be written as \( \vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 \) for some constants \( c_1 \) and \( c_2 \). But then we have that

\[
A\vec{x}_0 = A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2 = \lambda (c_1 \vec{v}_1 + c_2 \vec{v}_2) = \lambda \vec{x}_0.
\]

Thus \( \vec{x}_0 \) is an eigenvector as well. This implies the following phase portraits corresponding to \( \lambda < 0 \) and \( \lambda > 0 \).
(2) There is only one eigenvector $\vec{v}$. In this case we have a degenerate node. In general, and as seen before, when we have two eigenvalues and two linearly independent eigenvectors, then as $t \to \infty$, the trajectories approach the origin tangent to the slower decaying direction and as $t \to -\infty$, the trajectories approach infinity parallel to the faster decaying direction.

If we only have one eigenvector, then it serves as both the slow and the fast directions, which gives us a phase diagram like

In this case the independent solutions are given by

$$\vec{x}_1(t) = c_1 e^{\lambda t} \vec{v}, \quad \vec{x}_2(t) = c_2 t e^{-t} \vec{v}. $$

Dynamical Systems 9-26-05

### Classification of Fixed Points

Recall that the trace of the matrix $A$ was given by $\tau = \lambda_1 + \lambda_2$, and the determinant of $A$ was given by $\Delta = \lambda_1 \lambda_2$. Thus the characteristic equation is $\lambda^2 - \tau \lambda + \Delta = 0$. The quadratic formula then gives that

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. $$

From what we have seen up to this point, we know that if $\Delta = 0$, then at least one of the eigenvalues is zero, so the origin is a non-isolated fixed point, since there is a line or plane of fixed points. In the case $\Delta < 0$, the eigenvalues are real and have opposite sign (since $\Delta < 0$ and $\Delta = \lambda_1 \lambda_2$). This tells us that the origin is a saddle point in this situation. If $\Delta$ is positive, there are several cases that may occur, depending on the discriminant of the characteristic equation.

1. If $\Delta > 0$ and $\tau^2 - 4\Delta > 0$, then the eigenvalues are real and of the same sign. This implies that we have a node. The node will be stable if $\tau < 0$ and unstable if $\tau > 0.
2. If $\Delta > 0$ and $\tau^2 - 4\Delta < 0$, then $\lambda_1$ and $\lambda_2$ are complex. If $\tau < 0$, then we have a stable spiral, and if $\tau > 0$, we have an unstable spiral. If $\tau = 0$, then we have a center.

3. If $\Delta > 0$ and $\tau^2 - 4\Delta = 0$, then we lie on the border between nodes and spirals. We can have either star nodes or degenerate nodes, depending on the geometric multiplicity of the eigenvalue.

These results are summarized in this graph

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**Romeo and Juliet Problem**

Let $R(t)$ be Romeo’s feelings for Juliet as a function of time. Positive values signify love, and negative values signify hate. Similarly, let $J(t)$ denote Juliet’s feelings for Romeo. This model is known as the cautious lover’s model. We introduce coefficients $a < 0$ and $b > 0$, where $a$ is a measure of caution and $b$ is a measure of response. The model is

$$
\begin{align*}
\dot{R} &= aR + bJ \\
\dot{J} &= bR + aJ.
\end{align*}
$$

In words, when one of the lovers feels him or herself falling in love, he or she becomes cautious. But if the other partner begins to respond, there is an increase in interest. The system can be written in matrix form as

$$
\begin{pmatrix}
\dot{R} \\
\dot{J}
\end{pmatrix} =
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
\begin{pmatrix}
R \\
J
\end{pmatrix}.
$$

Here we have $\tau = 2a < 0$, $\Delta = a^2 - b^2$, and $\tau^2 - 4\Delta = 4b^2 > 0$. Also, we have a fixed point at $[R \ J]^T = [0 \ 0]^T$. There are two cases: (1) If $a^2 < b^2$, then $\Delta < 0$ and we have a saddle. The eigenvalues are $\lambda_1 = a + b$ and $\lambda_2 = a - b$ with corresponding eigenvectors $\vec{v}_1 = [1 \ 1]^T$ and $\vec{v}_2 = [1 \ -1]^T$. This gives us the following phase portrait:

(2) If $a^2 > b^2$, then both eigenvalues are negative, giving us a stable node.
Nonlinear Systems

Consider the system
\[
\begin{align*}
\dot{x} &= x + e^{-y} \\
\dot{y} &= -y
\end{align*}
\]

If \(-y = 0\), then \(y = 0\). Then \(x = -e^0 = -1\). Thus the fixed point is at \((x^*, y^*) = (-1, 0)\). The behavior near the fixed point can be determined by setting
\[
\begin{align*}
x &= -1 + u \\
y &= 0 + v
\end{align*}
\]
We can then derive an equation for \(u\) by noting that
\[
\dot{u} \frac{d}{dt}(x+1) = \dot{x} = x = e^{-y} = (-1+u) + (1 - y + O(y^2)) = -1 + u + 1 - v = u - v
\]
where the higher order terms have been dropped. Similarly, we can derive an equation for \(v\),
\[
\dot{v} = -v
\]
Thus the linearization about the fixed point is
\[
\begin{align*}
\dot{u} &= u - v \\
\dot{v} &= -v
\end{align*}
\]
Thus the matrix for the system is
\[
\begin{pmatrix}
1 & -1 \\
0 & -1
\end{pmatrix},
\]
with \(\tau = 0\) and \(\Delta = -1\), and \(\lambda_{1,2} = \pm 1\). The corresponding eigenvectors are
\[
\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]
The nullclines of the system are the curves defined by \(\dot{x} = 0\) and \(\dot{y} = 0\), or, in other words, where the flow is either entirely in the vertical direction or entirely in the horizontal direction. For the system in question, the nullclines are given by \(y = 0\) and \(y = -\ln(-x)\). Using the nullclines as guidelines, we will be able to get a good idea of how the phase portrait should look.
Linearization

Given the system
\[
\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{align*}
\]
let \((x^*, y^*)\) be a fixed point, so that \(f(x^*, y^*) = 0\) and \(g(x^*, y^*) = 0\). Let \(u = x - x^*\) and \(v = y - y^*\) be small perturbations from the fixed points. We wish to derive an equation for \(u\) and \(v\). We see that
\[
\dot{u} = \frac{d}{dt} u = \dot{x} = f(x, y) = f(x^*, y^*) + u \frac{\partial f}{\partial x}(x^*, y^*) + v \frac{\partial f}{\partial y}(x^*, y^*) + O(x^2 + y^2).
\]
Dropping the higher order terms, and recalling that \((x^*, y^*)\) is a fixed point, we have that
\[
\dot{u} = u \frac{\partial f}{\partial x}(x^*, y^*) + v \frac{\partial f}{\partial y}(x^*, y^*).
\]
Similarly, we derive an equation for \(v\) as
\[
\dot{v} = \frac{d}{dt} (y - y^*) = \dot{y} = g(x, y) = g(x^*, y^*) + u \frac{\partial g}{\partial x}(x^*, y^*) + v \frac{\partial g}{\partial y}(x^*, y^*) + O(x^2 + y^2),
\]
which gives us that
\[
\dot{v} = u \frac{\partial g}{\partial x}(x^*, y^*) + v \frac{\partial g}{\partial y}(x^*, y^*). 
\]
The matrix
\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]
is called the Jacobian matrix at \((x^*, y^*)\). The linearized system about \((x^*, y^*)\) is then given by
\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}.
\]
The question then arises: Does the linearized system correctly describe the behavior of the solution to the original system? Can we neglect the nonlinear terms? The answer is yes, if the linearization of the fixed point \((x^*, y^*)\) is not of borderline type. This means that the answer is yes for saddles, nodes, and spirals, but possibly no for centers, stars, and degenerate nodes. If the Jacobian matrix has \(\lambda_{1,2}\) with \(\Re \lambda_{1,2} \neq 0\), then the fixed point is called hyperbolic, meaning that the stability type of the fixed point is not affected by the nonlinear terms. A result of Hartman and Grebmanth tells us that if the Jacobian matrix \(J\) at a fixed point has no eigenvalues with zero real part, then there is a homeomorphism (defined in some neighborhood of the fixed point \((x^*, y^*)\)) that maps trajectories of the nonlinear system to the trajectories of the linearized system.

Example: Consider the system
\[
\begin{align*}
\dot{x} &= -x + x^3 \\
\dot{y} &= -2y
\end{align*}
\]
The fixed points are at (0, 0) and (±1, 0). The Jacobian matrix is given by

\[ A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix}. \]

Evaluating at (0, 0), we find that

\[ A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \]

with eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \), so the origin is a stable node. If we evaluate the Jacobian at (±1, 0), we get

\[ A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \]

with eigenvalues \( \lambda_1 = 2 \) and \( \lambda_2 = -2 \), so these fixed points are saddles. Because this system is decoupled, we can check our result by solving the two independent first order systems. In the \( y \) direction, we have exponential decay

In the \( x \) direction, we have \( \dot{x} = -x + x^3 \), so the lines \( x = 0 \), \( x = 1 \), and \( x = -1 \) are invariant. Putting this information together, we obtain the following phase portrait.

**Example**: Consider the system

\[
\begin{align*}
\dot{x} &= -y + ax(x^2 + y^2) \\
\dot{y} &= x + ay(x^2 + y^2)
\end{align*}
\]

The linearized system incorrectly predicts a center at the origin. The linearized system has matrix

\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]

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which has $\tau = 0$, $\Delta = 1$, and $\tau^2 - 4\Delta < 0$. This implies that the eigenvalues have the form $\lambda_{1,2} = \pm i\omega$ for some real number $\omega$. This tells us that $(0,0)$ should be a center. Upon analyzing the nonlinear system, we find that this is incorrect.

We introduce polar coordinates using the equations $x = r \cos \theta$, $y = r \sin \theta$ and $x^2 + y^2 = r^2$. Differentiating the last of these with respect to $t$, we find that

$$x\dot{x} + y\dot{y} = r\dot{r}.$$ 

Substituting our equations for $\dot{x}$ and $\dot{y}$, we find that

$$r\dot{r} = x(-y + ax(x^2 + y^2)) + y(x + ay(x^2 + y^2)) = ax^2(x^2 + y^2) + ay^2(x^2 + y^2) = a(x^2 + y^2)(x^2 + y^2) = ar^4.$$ 

This gives us that $\dot{r} = ar^3$. We can also derive an equation for $\theta$ by differentiating $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. This gives us that

$$\dot{\theta} = \frac{1}{r^2}[x(x + ay(x^2 + y^2)) - y(-y + ax(x^2 + y^2))].$$ 

$$= \frac{1}{r^2}[x^2 + axy(x^2 + y^2) + y^2 - axy(x^2 + y^2)] = \frac{x^2 + y^2}{r^2} = 1.$$ 

Thus, in polar coordinates our equations are

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}.$$ 

All trajectories rotate around the origin with constant angular velocity $\dot{\theta} = 1$. If $a < 0$, then $r(t) \to 0$ as $t \to \infty$. As $r \to 0$, the decay slows down. If $a = 0$, then indeed we have a center, and if $a > 0$, then we have an unstable spiral.

Robust cases are those that do not change under the effects of small perturbations. Examples are stable and unstable nodes and spirals and saddles. Marginal cases are those that have at least one eigenvalue with zero real part. These are centers, degenerate nodes, and non-isolated fixed points. The smallest perturbation will "knock" them into a different type of behavior. (See diagram from previous installment)

**Theorem 2 (Existence and Uniqueness)** Suppose that $\vec{f}$ is continuous and its partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous in an open, connected set $D$ in $\mathbb{R}^N$.

Then for any $\vec{x}_0 \in D$, there exist $\tau > 0$ such that the initial value problem $\vec{x} = \vec{f}(\vec{x})$, $\vec{x}(0) = \vec{x}_0$ has a unique solution $\vec{x}(t)$ for $t \in (-\tau, \tau)$.

We have the following consequences:

1. Different trajectories never intersect. This would imply that the vector field defined by $\vec{f}$ was not single-valued at some point.

2. If there is a closed orbit $C$, (in 2 dimensions), then any trajectory that starts inside $C$ is trapped inside $C$ for all time.
3. The Poincaré-Bendixon theorem: If a trajectory starts in $C$ and if there are no fixed points in the interior of $C$, then any trajectory that starts within $C$ must approach $C$ or approach another closed orbit within $C$.

The basic consequence of the Poincaré-Bendixon theorem is that if a trajectory is contained within a bounded region, then it is either

- a fixed point
- approaching a fixed point
- a closed orbit
- a limit cycle (approaching a closed orbit)

Also, this tells us that there can be no chaos in two dimensions.

**Index Theory**

Is there always a fixed point inside a closed trajectory? What types of fixed points can a particular system have? What types of fixed points can coalesce in a bifurcation? These are some of the questions that can be answered using index theory. Consider the system $\vec{x} = \vec{f}(\vec{x})$ in two dimensions. We consider any simple closed curve in the plane, not necessarily a trajectory, that does not pass through any fixed points, i.e. $\vec{f}(\vec{x}) \neq \vec{0}$ along $C$. We move the point $\vec{x}$ along $C$ in a counter-clockwise direction. The direction of $\vec{f}(\vec{x})$ is continuously changing. Let $[\varphi]_C$ be the total angle by which $\vec{f}(\vec{x})$ is turned as $\vec{x}$ returns to its original position. The index of the closed curve $C$ is

$$I_C = \frac{1}{2\pi}[\varphi]_C.$$  

This is the net number of counterclockwise revolutions of the vector $\vec{f}(\vec{x})$.

**Example**: Consider the system

$$\begin{cases} 
  \dot{x} = x \\
  \dot{y} = y 
\end{cases}.$$  

Here

$$\vec{f}(\vec{x}) = \begin{pmatrix} x \\ y \end{pmatrix}$$
and \( \varphi_C = 2\pi \), so \( I_C = \frac{1}{2\pi} \varphi_C = 1 \).

**Example**: Consider the system
\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= -y
\end{align*}
\]
Then \( \varphi_C = 2\pi \) and \( I_C = \frac{1}{2\pi} \varphi_C = 1 \) once again. Thus we see that reversing the arrows does not change the index of a curve.

**Example**: The system
\[
\begin{align*}
\dot{x} &= 1 \\
\dot{y} &= 2
\end{align*}
\]
has a constant vector field, and so \( \varphi_C = 0 \Rightarrow I_C = 0 \). Notice that there are no fixed points.

**Example**: For the last example, consider
\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= y
\end{align*}
\]
Here we have \( \varphi_C = -2\pi \), so \( I_C = -1 \).

**Properties of the Index**

1. If \( C \) is continuously deformed to \( C' \), without passing through any fixed points, then \( I_C = I_{C'} \). If there are no fixed points, then the index varies continuously as \( C \) is deformed to \( C' \), but \( I_C \) is integer valued, and if an integer valued function is continuous, it is constant.

2. If \( C \) does not enclose any fixed points, then its index is zero. In the absence of fixed points, we can shrink \( C \) to an arbitrarily small circle \( C' \). Since the vector field is continuous, we can make \( \varphi_{C'} \) arbitrarily close to 0, so \( I_C = I_{C'} = 0 \).

3. If we replace \( t \) with \(-t\), then \( \ddot{x} = \tilde{f}(\tilde{x}) \) becomes \( \ddot{x} = -\tilde{f}(\tilde{x}) \), which simply reverses all the arrows. This does not change the index, because the net change in the angle of the arrows is not changed. This is like adding \( \pi \) to \( \varphi \) everywhere. In this case we have \( I'_{C} = \frac{1}{2\pi} [\varphi + 2\pi]_{C} = \frac{1}{2\pi} [\varphi]_{C} = I_{C} \).

Thus the index is not related to stability.
4. If $C$ is actually a trajectory (a closed orbit), then $I_C = 1$. If $C$ is a trajectory, then the vector field is tangential to $C$, and if the angle of inclination of the tangent vector were to change by more than $2\pi$ as the trajectory was traversed, then the trajectory would have to intersect itself in a non-tangential way, which contradicts the single-valuedness of the vector field.

Dynamical Systems 10-5-05

Suppose that $x^*$ is an isolated fixed point. Then we can define $I_{x^*}$ to be equal to $I_C$ for any simple closed curve $C$ that encloses $x^*$ and no other fixed points. For a stable or unstable node we have $I_{x^*} = 1$, for any spiral we have $I_{x^*} = 1$, and for a saddle we have $I_{x^*} = -1$.

Theorem 3 If a closed curve $C$ encloses $n$ isolated fixed points $x_1^*, x_2^*, \ldots, x_n^*$, then $I_C = I_1 + I_2 + \cdots + I_n$, where $I_k$ is the index of $x_k^*$.

Proof: We can deform $C$ to $\Gamma$ as shown below so that each fixed point is contained in a small circle, and the small circles are connected by cross-cuts. We then have that

$$I_\Gamma = \frac{1}{2\pi} [\varphi]_\Gamma = \frac{1}{2\pi} \sum_{\text{cross-cuts}} [\varphi] + \frac{1}{2\pi} \sum_k [\varphi]_{\gamma_k}.$$  

As the width of the cross-cuts goes to zero, this becomes

$$I_\Gamma = \sum_k I_{x_k^*}.$$
Corollary 1 Any closed orbit in the plane must enclose fixed points whose indices sum to 1.

Corollary 2 If a closed orbit in the phase plane encloses $S$ saddles, $N$ nodes, $F$ spirals, and $C$ centers, then

$$S = N + F + C - 1.$$ 

Consider the complex vector field $\dot{z} = z^k$. If we write $z = x + iy$, then $\dot{x} + i\dot{y} = (x + iy)^k = r^k e^{ik\theta} \Rightarrow \dot{x} = r^k \cos k\theta$, $\dot{y} = r^k \sin k\theta$. If we let $k = 1$, and $C$ be the unit circle, then we have that $I_C = 1$. If $k = 2$, then the vector completes two revolutions as $C$ is traversed, so $I_C = 2$. For general $k$, the vector will complete $k$ revolutions as $C$ is traversed, so

$$I_C = \frac{1}{2\pi} [\varphi]_C = \frac{1}{2\pi} [k\theta]_C = k \frac{1}{2\pi} [\theta]_C = k$$

Now consider the vector field given by $\dot{z} = \bar{z}^k$. If $k = 1$, then $\dot{x} = r\cos \theta$, $\dot{y} = -r\sin \theta$, and as the unit circle is traversed, we see that $I_C = -1$. In the case $k = 2$, there are two clockwise revolutions made, so $I_C = -2$. Generally, $\varphi = -k\theta$, so

$$I_C = \frac{1}{2\pi} [-k\theta]_C = -k \frac{1}{2\pi} [\theta]_C = -k.$$ 

Suppose that we have a system

$$\begin{cases} 
\dot{x} = f(x, y) \\
\dot{y} = g(x, y)
\end{cases}.$$ 

We have that $\varphi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$, so to find the index, we will (1) find $\frac{d\varphi}{ds}$, where $s$ represents arclength, (2) Integrate the result with respect to $s$,

$$I_C = \frac{1}{2\pi} \oint_C d\varphi.$$ 

Since

$$\tan \varphi = \frac{\dot{y}}{\dot{x}} = \frac{g(x, y)}{f(x, y)},$$

we have

$$\frac{d}{ds} \tan \varphi = \frac{d}{ds} \frac{g}{f}.$$
and upon differentiating both sides with respect to \(s\), we find that
\[
\frac{1}{\cos^2 \varphi} \frac{d\varphi}{ds} = \frac{df - df}{f^2}. 
\]
This then gives us that
\[
\frac{d\varphi}{\cos^2 \varphi} = \frac{dg - df}{f^2}. 
\]
Since
\[
\cos^2 = \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 = \left(\frac{f}{\sqrt{f^2 + g^2}}\right)^2 = \frac{f^2}{f^2 + g^2},
\]
we have that
\[
\frac{(f^2 + g^2) d\varphi}{f^2} = \frac{dg - df}{f^2} \Rightarrow d\varphi = \frac{dg - df}{f^2 + g^2} \Rightarrow I_C = \frac{1}{2\pi} \oint d\varphi = \int_C \frac{dg - df}{f^2 + g^2}.
\]

We will study the Lotka-Volterra model of competition. In particular, we will consider a population of sheep competing with a population of rabbits for a limited resource, which in this case would be grass. Let \(x(t)\) be the population of rabbits and \(y(t)\) be the population of sheep. We assume the following:

1. Each population grows logistically in the absence of the other, and the rabbits have a higher growth rate.

2. The rate of conflict for food is proportional to the size of the population. We also assume that these conflicts reduce the growth rate more for the rabbits, as they are less resistant to famine.

Our system is given by
\[
\begin{align*}
\dot{x} &= x(3 - x - 2y) \\
\dot{y} &= y(2 - x - y),
\end{align*}
\]
where \(x \geq 0\) and \(y \geq 0\). The first term in the parenthesis is the growth rate for each species and the other two terms represent competition for food. The first step is to find the fixed points. Setting each equation equal to zero, we get
\[
\begin{align*}
3 - x - 2y &= 0 \text{ or } x = 0 \\
2 - x - y &= 0 \text{ or } y = 0.
\end{align*}
\]
This gives us fixed points at \((0, 0)\), \((0, 2)\), \((3, 0)\), and \((1, 1)\). Next, we find the Jacobian matrix. This turns out to be
\[
J = \begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{pmatrix} = \begin{pmatrix}
3 - 2x - 2y & -2x \\
-y & 2 - x - 2y
\end{pmatrix}.
\]
The eigenvalues of \( J \) vary according to the fixed point at which we evaluate. At the fixed point \((0, 0)\), we have that
\[
J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \lambda_1 = 3, \quad \lambda_2 = 2, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Thus, the origin is an unstable node. The trajectories leave the origin parallel to the eigenvector \( \vec{v}_2 \). At the fixed point \((0, 2)\), we have
\[
J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \quad \tau = -3, \quad \Delta = 2, \quad \lambda_1 = -1, \quad \lambda_2 = -2.
\]
Trajectories approach this stable fixed point along the slower eigendirection (corresponding to \( \lambda_1 \)). To find an eigenvector, we must solve the system
\[
\begin{pmatrix} 0 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2p - q = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\]
At the fixed point \((3, 0)\), we have that
\[
J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}, \quad \tau = -4, \quad \Delta = 3, \quad \lambda_1 = -3, \quad \lambda_2 = -1.
\]
Thus we have a stable node, and the trajectories approach the origin parallel to the slower decaying eigendirection, which is parallel to
\[
\vec{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.
\]
Finally, at the fixed point \((1, 1)\) we have
\[
J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \quad \tau = -2, \quad \Delta = -1.
\]
The eigenvalues are given by
\[
\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) = -1 \pm \sqrt{2}.
\]
Since \( \lambda_1 = -1 + \sqrt{2} > 0 \) and \( \lambda_2 = -1 - \sqrt{2} < 0 \), we have a saddle point. To find and eigenvector corresponding to \( \lambda_1 \), we solve the system
\[
\begin{pmatrix} -\sqrt{2} & -2 \\ -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-\sqrt{2}}{2} \end{pmatrix}.
\]
Similarly, we find that an eigenvector corresponding to \( \lambda_2 \) is
\[
\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}.
\]
We also note that $\dot{x} = 0$ for $x = 0$ and $\dot{y} = 0$ for $y = 0$. This implies that these nullclines are actually trajectories of the system. Putting all this together, we find the following phase portrait:

Here we see a demonstration of the principle of competitive exclusion: Two species competing for the same limited resource cannot coexist.

**Definition 4** Given an attracting fixed point $\bar{x}^*$, the basin of attraction of $\bar{x}^*$ is the set of initial conditions $\bar{x}_0$ such that $\bar{x}(t) \to \bar{x}^*$ as $t \to \infty$.

In this model, we see that closed orbits are impossible. The stable and unstable nodes have index 1, and the saddle has index -1. Also, $x = 0$ and $y = 0$ are trajectories. Since trajectories cannot intersect, a closed orbit would have to exist entirely within the first quadrant. But a closed orbit has index 1. No closed orbit can enclose any of the nodes, as they lie on vertical and horizontal trajectories. That leaves only the saddle, but it has the wrong index. Thus there can be no closed paths.

**Dynamical Systems 10-12-05**

**Conservative Systems** Newton’s laws of motion tell us that

$$m\ddot{x} = F(x).$$

If the function $F$ does not depend on $\dot{x}$ or $t$, then we can define a potential energy function for the system via

$$V(x) = -\int F(x) \, dx \Rightarrow F(x) = -\frac{dV}{dx}.$$ 

**Definition 5** A real valued function $E(\bar{x}, \dot{\bar{x}})$ that is constant on all trajectories (but not constant on any open set) is called a conserved quantity of the system. In particular, we have

$$E(\bar{x}, \dot{\bar{x}}) = k$$

along a trajectory.

By the definition of the potential energy function, we have that

$$m\ddot{x} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} + V'(x)\dot{x} = 0 \Rightarrow \frac{d}{dt}\left[ \frac{m\dot{x}^2}{2} + V(x) \right] = 0.$$
Thus the quantity 
\[ \frac{m\dot{x}^2}{2} + V(x) \]
is constant for all \( t \).

A conservative system cannot have an attracting fixed point. Indeed, if the opposite were true, then there would be a \( \delta > 0 \) such that \( |\vec{x}_0 - \vec{x}^*| < \delta \) would imply that \( \vec{x}(t) \to \vec{x}^* \) as \( t \to \infty \). Then in particular we have that \( E(\vec{x}) = (\vec{x}^*) \) for any \( \vec{x} \) such that \( |\vec{x} - \vec{x}^*| < \delta \), but the definition of \( E(\vec{x}) \) tells us that \( E(\vec{x}) \) is not constant on any open set.

**Example**: Consider the potential function 
\[ V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \]
The corresponding equation is 
\[ m\ddot{x} = F(x) = -V'(x) = x - x^3. \]
We assume that \( m = 1 \) and let \( \dot{x} = y \). Then we have the system
\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= x - x^3.
\end{aligned}
\]
A graph of \( V(x) \) looks like

If we look at the linearization near the fixed points \((0, 0)\) and \((\pm 1, 0)\), we find that at \((0, 0)\), the eigenvalues are \( \pm 1 \) and corresponding to the eigenvalue +1, we have eigenvector
\[ \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
According to the Grobman-Hartman theorem, the phase portrait of the non-linear system near \((0, 0)\) is topologically equivalent to the phase portrait of the phase portrait of the linear system. About the fixed point \((1, 0)\), we find eigenvalues \( \pm i\sqrt{2} \), which would indicate a center. However, since the real part of the eigenvalues is zero, the Grobman-Hartman theorem does not apply. Recall the previous example of the system
\[
\begin{aligned}
\dot{x} &= y + ax(x^2 + y^2) \\
\dot{y} &= x + ay(x^2 + y^2).
\end{aligned}
\]
where the linearization predicted a center, but the actual system (when converted to polar coordinates) was obviously a spiral. In this example, the nonlinear system will actually have centers at \((\pm 1, 0)\) because of the conserved energy. We have that
\[
E = \frac{m\ddot{x}^2}{2} + V(x) = \frac{y}{2} - \frac{x^2}{2} - \frac{x^4}{4} = \text{constant}.
\]
The trajectories are closed curves defined by the contour lines or level curves of constant energy.

We have several types of motion:

1. Oscillations at the bottom of one potential well.
2. Large oscillations that repeatedly take the particle "over the hump".
3. Homoclinic orbits that start and end at the same fixed point.

Such a motion is called an undamped particle in a double potential well.

**Theorem 4** Suppose that for the system
\[
\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{x} \in \mathbb{R}^2
\]
has a conserved quantity \(E(\vec{x})\), and that an isolated fixed point \(\vec{x}^*\) is a local minimum of \(E(\vec{x})\). Then \(\vec{x}^*\) is a center. Indeed, all trajectories are contour lines of \(E(\vec{x})\).

Since \(\vec{x}^*\) is a minimum, the lines are closed near \(\vec{x}^*\). The trajectories cannot stop at some other fixed point because \(\vec{x}^*\) is isolated, so they must go around \(\vec{x}^*\), giving us a center.

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**Hamiltonian Systems**

Suppose that \(H(p, q)\) is a smooth, real-valued function of two variables \(p\) and \(q\). The variable \(p\) is called the conjugate momentum, and \(q\) is called a generalized coordinate. A Hamiltonian system is one that has the form
\[
\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial q}
\end{align*}
\]
for some $H(p, q)$. For example, a harmonic oscillator of mass $m$, spring constant $k$, displacement $x$ and momentum $p$ has Hamiltonian function

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}.$$  

Then

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x},$$

which implies that $p = m\dot{x}$. We also have that

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial H}{\partial x} = -\frac{2kx}{2} = -kx = m\ddot{x}.$$  

Thus $m\ddot{x} + kx = 0$. Also we see that

$$\frac{\partial}{\partial t} H = \frac{\partial}{\partial t} \left( \frac{p^2}{2m} + \frac{kx^2}{2} \right) = \frac{2p\dot{p}}{2m} + \frac{2kx\dot{x}}{2} = \frac{p\dot{p}}{m} + kx\ddot{x}.$$  

Substituting $\dot{x} = \frac{p}{m}$ and $\dot{p} = -kx$, this becomes

$$\frac{p}{m}(-kx) + kx\left( \frac{p}{m} \right) = 0.$$  

Thus the Hamiltonian is a conserved quantity. In any Hamiltonian system it is true that $H$ is a conserved quantity, and the trajectories are contour lines of $H(p, q) = C$, since

$$\frac{d}{dt} H(x, p) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \left( -\frac{\partial H}{\partial x} \right) = 0.$$  

**Limit Cycles**

A limit cycle is an isolated closed trajectory. Limit cycles come in several varieties, namely stable, unstable, and half-stable.

The presence of a limit cycle tells us automatically that neighboring trajectories are not closed nor are they attracting. In a linear system, a closed trajectory cannot be isolated, since if $x(t)$ is a closed orbit, then $Cx(t)$ is also a solution to the system that is a closed orbit.

**Example**: Consider the system

$$\begin{cases} 
\dot{r} = r(-r^2) \\
\dot{\theta} = 1
\end{cases}$$  

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for \( r \geq 0 \). Since the radial and angular dynamics are uncoupled, we can analyze them separately. In the \( \theta \) direction, the motion is rotation with unit angular velocity. As for \( r \), we have

Thus the motion settles down to sinusoidal oscillations.

**Example**: Consider the Van der Pol oscillator, whose equation of motion is

\[
\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu \geq 0.
\]

The middle term on the right hand side is a nonlinear damping term. If \( |x| > 1 \), then we have positive damping and large oscillations decay. If \( |x| < 1 \), then there is negative damping, and small oscillations grow. Thus there exists a unique limit cycle for any \( \mu > 0 \).

There are three ways to rule out the existence of closed orbits, the first of which is to check if the system is a gradient system of the form

\[
\dot{x} = -\nabla v,
\]

where \( v \) is a continuously differentiable single-valued scalar function (think of \( v \) as a potential). Then we can use

**Theorem 5** *Closed orbits are impossible in a gradient system.*

**Proof**: Suppose that there did exist a closed orbit. Consider the change in \( v \) after one circle. We must have \( \Delta v = 0 \), since \( v \) is a single-valued function. But

\[
\Delta v = \int_0^T \frac{dV}{dt} \, dt = \int_0^T \nabla v \cdot \dot{x} \, dt = \int_0^T (\nabla v \cdot \dot{x}) \, dt = \int_0^T -|\dot{x}|^2 \, dt,
\]

so \( \dot{x} = 0 \). Thus the closed orbit is actually a fixed point.
Last time we proved that closed orbits are impossible in a gradient system, i.e. a system such that
\[ \dot{x} = -\nabla V, \]
where \( V \) is a potential function.

**Example**: Consider the system
\[
\begin{align*}
\dot{x} &= \sin y \\
\dot{y} &= x \cos y.
\end{align*}
\]
If we take \( V(x, y) = -x \sin y \), then
\[
-\frac{\partial V}{\partial y} = x \cos y, \quad -\frac{\partial V}{\partial x} = \sin y,
\]
so this is a gradient system.

The second type of system that we can guarantee has no closed orbits is a system which has a Liapunov function \( V(x) \). Suppose that \( \dot{x} = \vec{f}(\vec{x}) \) and that \( \vec{x}^\ast \) is a fixed point of the system. If \( V(x) \) is a continuously differentiable, real valued function such that
(a) \( V(x) > 0 \) for every \( x \neq x^\ast \), and \( V(x^\ast) = 0 \).
(b) \( V(x) < 0 \) for every \( x \neq x^\ast \), meaning that all trajectories flow "downhill".

**Example**: There are no periodic solutions to the nonlinearly damped oscillator
\[
\ddot{x} + (\dot{x})^3 + x = 0.
\]
Consider the "energy-like" function \( E(x, \dot{x}) = (1/2)(x^2 + \dot{x}^2) \). If there is a periodic solution, then after one cycle, \( x \) and \( \dot{x} \) would have to return to their starting values so that \( \Delta E = 0 \) around any closed orbit. But
\[
\Delta E = \int_0^t \frac{dE}{dt} \, dt = \int_0^t \dot{E} \, dt = \int_0^t \frac{d}{dt} \left( \frac{1}{2} x^2 + \frac{1}{2} \dot{x}^2 \right) \, dt \\
= \int_0^t (x \ddot{x} + \dot{x} \dot{x}) \, dt = \int_0^t \dot{x}^2 + \ddot{x} \, dt = \int_0^t -\dot{x} \ddot{x}^3 = -\int_0^t \dot{x}^4 \leq 0.
\]
We have equality in the above only if \( \dot{x} = 0 \) for every \( t \), which would be a fixed point and not a closed orbit.

**Example**: A system with a Liapunov function is
\[
\begin{align*}
\dot{x} &= -x + 4y \\
\dot{y} &= -x - y^3.
\end{align*}
\]
Let \( V(x, y) = x^2 + a y^2 \), with \( a \) to be determined later. Then
\[
\frac{d}{dt} V = 2x \dot{x} = 2ay \dot{y} = 2x(-x + 4y) + 2ay(-x - y^3) \\
= -2x^2 + 8xy - 2axy - 2ay^4 = -2x^2(8 - 2a)xy - 2a^4.
\]
If we choose $a = 4$, then this becomes 

$$-2x^2 - 8y^4.$$ 

We then have that  

$$V(x, y) = x^2 + 4y^2 > 0, \quad \dot{V}(x, y) = -2x^2 - 8y^4 < 0$$ 

for all $(x, y) \neq (0, 0)$, so this is a Liapunov function. This guarantees that there are no closed orbits just as in the previous example.

Our third method for ruling out our closed orbits is called Dulac’s Criterion, which is based on Green’s Theorem.

**Theorem 6** Suppose that  

$$\dot{\vec{x}} = \vec{f}(\vec{x}),$$ 

where $\vec{f}$ is a continuously differentiable vector field defined on a simply connected region $R$. If there exists a continuously differentiable, real-valued function $g(x)$ such that $\text{div}(g(x)\dot{\vec{x}})$ has one sign in $R$, then the system does not have closed orbits in $R$.

**Proof**: Suppose that there does exist a closed orbit $C$ in $R$. Then Green’s Theorem tells us that if $A$ is the area enclosed by $C$, then 

$$0 \neq \int \int_A \text{div}(g(x)\dot{\vec{x}}) \, dx \, dy = \int_C g(x)\dot{\vec{x}} \cdot \vec{n} \, ds = 0.$$ 

The second integral is always zero since the vector field around the closed orbit is always orthogonal to the normal of the orbit.

**Example**: Consider the system 

$$\begin{cases} 
\dot{x} = y \\
\dot{y} = -x - y + x^2 + y^2
\end{cases}.$$ 

Take $g(x) = e^{-2x}$. Then  

$$\text{div}(g(x)\dot{x}) = \nabla \cdot \left( \frac{e^{-2x} \dot{x}}{e^{-2x} \dot{y}} \right) = \nabla \cdot \left( e^{-2x} \left( -x - y + x^2 + y^2 \right) \right)$$

$$= -2ye^{-2x} + e^{-2x}(-1) + 2ye^{-2x} = -e^{-2x} < 0.$$ 

To establish the existence of closed orbits, we use the Poincaré-Bendixon theorem.

**Theorem 7** Suppose that  

1. $R$ is a closed, bounded subset of $\mathbb{R}^2$,  
2. $\dot{\vec{x}} = \vec{f}(\vec{x})$, and $\vec{f}$ is a continuously differentiable vector field on an open set contained in $R$, 

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(3) $R$ does not contain any fixed points,

(4) There exists a trajectory $C$ that is “confined to $R$”, i.e. $C$ starts in $R$ and stays in $R$ as $t \to \infty$.

Then $R$ contains a closed orbit, and $C$ is either a closed orbit or it spirals toward the closed orbit as $t \to \infty$.

To show that $R$ has a confined trajectory, we choose $R$ to be a trapping region such that the vector field points inward on the boundary of $R$.

Example: The biochemical process of glycolysis (breaking down sugar) concerns the reactions of adenosine diphosphate (ADP) and fructose-6 phosphate (F6P). We will denote the concentrations of these chemicals by $x$ and $y$ respectively. The system that governs the reaction is

$$
\begin{align*}
\dot{x} &= -x + ay + x^2y \\
\dot{y} &= b - ay - x^2y,
\end{align*}
$$

where $a$ and $b$ are kinetic parameters. To locate a closed orbit, we construct a trapping region. First, we find the nullclines.

\dot{x} = 0 \Rightarrow -x + ay + x^2y = 0 \Rightarrow y = \frac{x}{a + x^2}.

Also, we note that

\dot{x} > 0 \Rightarrow -x + ay + x^2y > 0 \Rightarrow ay + x^2y > x \Rightarrow y > \frac{x}{a + x^2}.

This gives us a graph of the $x$ nullcline.

As for the $y$ nullcline,

\dot{y} = 0 \Rightarrow b - ay - x^2y = 0 \Rightarrow y = \frac{b}{a + x^2}.
\( \dot{y} > 0 \Rightarrow b - ay - x^2 y > 0 \Rightarrow y < \frac{b}{a + x^2} \).

This information gives us this graph of the \( y \) nullcline.

When we combine the information in these two graphs, we get the following:

The \( x \) and \( y \) axes seem to form two sides of a trapping region, so we just need to find the other side(s). We draw a horizontal line from point \( b/a \) on the \( y \) axis and then when that line is directly over point \( b \) on the \( x \) axis, we make the slope \(-1\), and then continue until we intersect the \( x \) nullcline, as shown below.

It then remains to show that trajectories cross the sloped line inwards. We do this by comparing \( \dot{x} \) to \(-\dot{y}\) on this line. In this region \( \dot{x} > 0 \) and \( \dot{y} < 0 \), so if \(-\dot{y} > \dot{x}\), then vectors near the sloped line will have slope less than \(-1\), and will therefore cross into the trapping region. We see that

\[
\dot{x} - (-\dot{y}) = x + ay + x^2 y + b - ay - x^2 y = b - x, \quad b - x < 0 \Rightarrow x > b.
\]

As this requirement is satisfied by our construction, we have a trapping region.

Next we must remove the fixed point from our region, and to ensure that the trapping region is still a trapping region, we must make sure that the fixed point is repelling. To this end, we take the Jacobian.

\[
J = \begin{pmatrix}
-1 + 2xy & a + x^2 \\
-2xy & -a - x^2
\end{pmatrix}
\]
The fixed point of the system is
\[ x^* = b, \quad y^* = \frac{b}{a + b^2}. \]
We will calculate \( \Delta \) and \( \tau \) for the system, and then find a value of \( b \) for which \( \tau, \Delta > 0 \), so that we will have an unstable node or spiral.

\[ J(x^*, y^*) = \begin{pmatrix} -1 + \frac{2b^2}{a + b^2} & a + b^2 \\ -\frac{2b^2}{a + b^2} & -(a + b^2) \end{pmatrix} \]

\[ \Rightarrow \Delta = a + b^2, \quad \tau = -1 + \frac{2b^2}{a + b^2} - (a + b^2) = \frac{-a + b^2 - a^2}{a + b^2}. \]

In order to have \( \tau > 0 \), we must have the numerator positive, so
\[ b^4 - b^2(1 - 2a) + (a^2 + a) < 0, \]
which is a quadratic in \( b^2 \) that is also concave up. Thus we want \( b \) to lie in the region between the roots of the quadratic. We have that
\[ b^2 = \frac{1 + 2a \pm \sqrt{(1 - 2a)^2 - 4(a^2 + a)}}{2} = \frac{1}{2}(1 - 2a \pm \sqrt{1 - 8a}). \]

This gives us the roots as a function of \( a \). A graph of this function looks like

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**Relaxation Oscillations** Consider the Van der Pol Equation
\[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu \gg 1. \]
Since \( \mu \) is large, we have a strongly nonlinear term. We can rewrite the equation in the form
\[ \frac{d}{dt} \left[ \mu \left( \frac{x^3}{3} - x \right) \right] + x = 0. \]
We let \( F(x) = x^3/3 - x \) and \( w = \dot{x} + \mu F \), and this gives us the system
\[
\begin{cases}
\dot{x} = w - \mu F(x) \\
\dot{w} = -x
\end{cases}
\]
Further, we may let \( w = \mu y \), and then the system becomes

\[
\begin{aligned}
\dot{x} &= \mu [y - F(x)] \\
\dot{y} &= -\frac{\dot{x}}{\mu}
\end{aligned}
\]

The \( \dot{x} = 0 \) nullcline is given by

\[ y = F(x) = \frac{1}{3} x^3 - x. \]

If \( \dot{x} \approx 0 \), then \( \dot{y} \sim \mathcal{O}(1/\mu) \), so we have slow buildup along the nullcline. Similarly, on the horizontal parts of the trajectory indicated in the diagram below, the velocity is \( \dot{x} \sim \mathcal{O}(\mu) \). Thus we have two separate time scales within the problem. The time to crawl up the slopes is \( \mathcal{O}(\mu) \) and the time to cross the horizontal lines is \( \mathcal{O}(1/\mu) \).

The trajectory will stay fairly close to the graph of \( F(x) \), but how close? Near the nullcline, \( \dot{x} \approx 0 \) and \( \dot{y} \) very small as well. Thus we may assume \( \dot{x} \approx \dot{y} \), and if we let \( h = y - F(x) \), then

\[ uh \sim \frac{1}{u} \Rightarrow h \sim \mathcal{O} \left( \frac{1}{\mu^2} \right) . \]

We are also interested in knowing the period of the oscillation. From our previous analysis, we conclude that the period \( T \) is about the time that the phase particle spends near the nullcline \( y = F(x) \). Thus

\[ T = 2 \int_{t_B}^{t_A} dt. \]

But \( y \approx F(x) \) near the nullclines, so

\[
\frac{dy}{dt} = F'(x) \dot{x} \Rightarrow -\frac{x}{\mu} = \dot{x} (x^2 - 1) \Rightarrow \frac{dx}{dt} (x^2 - 1) = -\frac{x}{\mu} \Rightarrow dt = -\frac{\mu (x^2 - 1)}{x} \ dx
\]

on the slow branches. Therefore

\[
T = 2\mu \int_{x_1}^{x_2} \left( x - \frac{1}{x} \right) \ dx = 2\mu \left( \frac{x^2}{2} - \ln x \right) \bigg|_{x_1}^{x_2=2} = 2\mu \left[ 2 - \ln 2 - \frac{1}{2} \right] = \mu (3 - 2\ln 2) = \mathcal{O}(\mu).
\]

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Weakly Nonlinear Oscillations

We will use perturbation theory to study equations of the type
\[ \ddot{x} + \omega^2 x = \epsilon h(x, \dot{x}), \quad \epsilon \ll 1. \]

When \( \epsilon = 0 \), this is just a harmonic oscillator with solution
\[ x = A \cos \omega t + B \sin \omega t. \]

When \( \epsilon \neq 0 \), we use perturbation theory to represent the solution in an asymptotic expansion
\[ x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots. \]  

(3)

Consider the system
\[
\begin{aligned}
\ddot{x} + x &= -2\epsilon \dot{x} \\
x(0) &= 0 \\
\dot{x}(0) &= 1
\end{aligned}
\]

Since \( \epsilon \) is small, this is a weakly damped oscillator. If, as usual, we let \( \dot{x} = y \), then the system becomes
\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -x - 2\epsilon y \\
x(0) &= 0 \\
y(0) &= 1
\end{aligned}
\]

\Rightarrow \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -2\epsilon \end{pmatrix}.

Upon examining the matrix \( A \), we find that \( \tau = -2\epsilon \) and \( \Delta = 1 \). This gives us eigenvalues
\[
\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4}\Delta}{2} = \frac{-2\epsilon \pm \sqrt{4\epsilon^2 - 1}}{2} = -\epsilon \pm \sqrt{\epsilon^2 - 1} = -\epsilon \pm i\sqrt{1-\epsilon^2}.
\]

Thus the general solution is
\[ x(t) = (C_1 \cos \sqrt{1-\epsilon^2} t + C_2 \sin \sqrt{1-\epsilon^2} t)e^{-\epsilon t}. \]

The initial condition \( x(0) = 0 \) implies that \( C_1 = 0 \), and \( \dot{x}(0) = 1 \) implies
\[
(C_2 \sqrt{1-\epsilon^2} \cos \sqrt{1-\epsilon^2} e^{-\epsilon t} + B \sin \sqrt{1-\epsilon^2} t(-\epsilon e^{-\epsilon t})) \bigg|_{t=0} = C_2 \sqrt{1-\epsilon^2} = 1 \quad \Rightarrow \quad C_2 = \frac{1}{\sqrt{1-\epsilon^2}}.
\]

The exact solution is
\[ x(t, \epsilon) = \frac{1}{\sqrt{1-\epsilon^2}} e^{-\epsilon t} \sin(\sqrt{1-\epsilon^2} t). \]

In our first attempt to apply perturbation theory to this problem, we will see that it fails. Substituting the expansion (3) into the equation, we obtain
\[
\frac{d^2}{dt^2} [x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots] + [x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots] = -2\epsilon \frac{d}{dt} [x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots].
\]

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Collecting powers of $\epsilon$, we obtain the following equations:

$\epsilon^0$:
$$\ddot{x} + x = 0,$$

$\epsilon^1$:
$$\ddot{x}_1 + x_1 = -2\dot{x}_0,$$

$\epsilon^2$:
$$\ddot{x}_2 + x_2 = -2\dot{x}_1.$$

Since the monomials in $\epsilon$ are linearly independent, the equations

$$x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \cdots = 0,$$

$$\dot{x}_0(0) + \epsilon \dot{x}_1(0) + \epsilon^2 \dot{x}_2(0) + \cdots = 1$$

imply

$$x_0(0) = 0, x_1(0) = 0, \ldots,$$

$$\dot{x}_0(0) = 1, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0, \ldots.$$

Applying the correct initial condition to the $\epsilon^0$ equation, we find that

$$x_0(t) = \sin t.$$

This makes the $\epsilon^1$ equation

$$\ddot{x}_1 + x_1 = -2 \cos t, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

The solution to this equation is $x_1(t) = -t \sin t$, which grows without bound as $t \to \infty$. This is because the forcing term in the $\epsilon^1$ equation is at the natural frequency of this equation, which causes resonant terms. As is, our perturbation solution is

$$x_{pert}(t, \epsilon) = x_0(t) = \epsilon x_1(t) + \epsilon^2 x_2(t) = \sin t - ct \sin t + \mathcal{O}(\epsilon^2),$$

If we expand the exact solution found above in Taylor Series, we find that

$$x_{ext}(t, \epsilon) = (1 - ct) \sin t + \mathcal{O}(\epsilon^2),$$

which coincides with the perturbation solution. However, qualitatively the behavior differs. As $t \to \infty$, $x_{ext}(t, \epsilon) \to 0$, but $x_{pert} \to \infty$. This implies that $x_{pert}$ is only a good approximation up for $ct \ll 1 \Rightarrow t \ll 1/\epsilon$, or $t \sim \mathcal{O}(1/\epsilon)$. As we noted above $x_{pert}$ contains a secular term, i.e. a term that grows without bound as $t \to \infty$. If we are interested in long term behavior, we must do something different.

We notice that in the solution

$$x(t, \epsilon) = \frac{1}{\sqrt{1 - \epsilon^2}} e^{-ct} \sin(\sqrt{1 - \epsilon^2} t),$$

there are two time scales: a fast time $t \sim \mathcal{O}(1)$ for the sinusoidal oscillations, and a slow time $t \sim \mathcal{O}(1/\epsilon)$ for amplitude decay. We let $\tau = t$ be the fast time and $T = \epsilon t$ be the slow time. We consider the two as independent variables, i.e. functions of the slow time $T$ will be considered as constants on the fast time scale and vice versa. With this approach, we have the expansion

$$x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + \mathcal{O}(\epsilon^2).$$
Recall that we are examining the equation
\[ \ddot{x} + 2\epsilon\dot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1, \]
and that we have introduced an asymptotic expansion
\[ x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + \mathcal{O}(\epsilon^2), \]
where we have a fast time \( \tau = t \) and a slow time \( T = \epsilon t \). We differentiate this expansion with respect to \( t \), obtaining
\[ \dot{x} = \frac{d}{dt} x = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T} \frac{dT}{dt} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T} = x_{\tau} + \epsilon x_T = \partial_\tau (x_0 + \epsilon x_1) + \epsilon \partial_T (x_0 + \epsilon x_1) + \mathcal{O}(\epsilon^2) \]
\[ = \partial_\tau x_0 + \epsilon (\partial_T x_0 + \partial_\tau x_1) + \mathcal{O}(\epsilon^2), \]
and
\[ \ddot{x} = (\partial_\tau + \epsilon \partial_T) \dot{x} = \partial_{\tau\tau} x_0 + \epsilon (\partial_{\tau T} x_0 + \partial_{\tau \tau} x_1) + \epsilon \partial_{T T} x_0 + \mathcal{O}(\epsilon^2) \]
\[ = \partial_{\tau\tau} x_0 + \epsilon \partial_{\tau} x_1 + 2\epsilon \partial_T x_0 + \mathcal{O}(\epsilon^2) = \partial_{\tau\tau} x_0 + \epsilon (\partial_{\tau} x_1 + 2\partial_T x_0) + \mathcal{O}(\epsilon^2). \]
Substituting these expressions into the original equation, it becomes
\[ \partial_{\tau\tau} x_0 + \epsilon (\partial_{\tau} x_1 + 2\partial_T x_0) + 2\epsilon \partial_{\tau} x_0 + x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2) = 0. \]
We then collect powers of \( \epsilon \), to obtain
\[ \epsilon^0: \quad \partial_{\tau\tau} x_0 + x_0 = 0, \]
which has solution \( x_0(\tau, T) = A(T) \sin \tau + B(T) \cos \tau \). Collecting terms of order \( \epsilon \) gives
\[ \epsilon^1: \quad \partial_{\tau\tau} x_1 + 2\partial_{\tau T} x_0 + 2\partial_T x_0 = x_0 = 0 \Rightarrow \quad \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - 2\partial_T x_0 = -2(\partial_{\tau T} x_0 + \partial_T x_0). \]
Performing the required differentiations on \( x_0 \) and substituting into the above equation, we get
\[ \partial_{\tau\tau} x_1 + x_1 = -2(A + A') \cos \tau + 2(B + B') \sin \tau, \]
where \( A' = dA/dT \) and \( B' = dB/dT \). To avoid secular terms, we set
\[ A + A' = 0 \Rightarrow A(T) = A(0)e^{-T}, \]
\[ B + B' = 0 \Rightarrow B(T) = B(0)e^{-T}. \]
We can find \( A(0) \) and \( B(0) \) via the initial conditions \( x(0) = 0, \ \dot{x}(0) = 1 \). We have
\[ 0 = x(0) = x_0(0, 0) + \epsilon x_1(0, 0) + \mathcal{O}(\epsilon^2) \Rightarrow x_0(0, 0) = 0, \quad x_1(0, 0) = 0, \]
and
\[ 1 = \dot{x}(0) = \frac{\partial}{\partial \tau} x_0(0,0) + \epsilon \frac{\partial}{\partial T} x_0(0,0) + \epsilon \frac{\partial}{\partial \tau} x_1(0,0) \]
\[ \Rightarrow \frac{\partial}{\partial \tau} x_0(0,0) = 1, \quad \frac{\partial}{\partial T} x_0(0,0) = 0, \quad \frac{\partial}{\partial \tau} x_1(0,0) = 0. \]
Since \( x_0 = A \sin \tau + B \cos \tau \), we must have that \( B(0) = 0 \), and \( A(0) = 1 \). This implies that
\[ x_0(\tau, T) = e^{-T} \sin \tau, \]
so
\[ x(t, \epsilon) = e^{-t} \sin \tau + O(\epsilon) = e^{-t} \sin t + O(\epsilon). \]
Now as \( t \to \infty \), we have \( x_{\text{pert}}(t) \to 0. \)

Dynamical Systems 10-28-05

Our task today is to show that the Van der Pol oscillator
\[ \ddot{x} + x + \epsilon (x^2 - 1) \dot{x} = 0 \]
has a stable limit cycle with radius \( 2 = O(\epsilon) \) and frequency \( 1 + O(\epsilon^2) \). We use the two-timing method, and collect powers of \( \epsilon \) to obtain the following equations:
\[ \epsilon^0 : \quad \partial_{\tau \tau} x_0 + x_0 = 0, \]
\[ \epsilon^1 : \quad \partial_{\tau \tau} x_1 + x_1 = -2 \partial_{\tau \tau} x_0 - (x_0^2 - 1) \partial_{\tau} x_0. \]
The first equation is that of a simple harmonic oscillator, which has solution
\[ x_0 = r(T) \cos(\tau + \varphi(T)). \]
Here \( r(T) \) and \( \varphi(T) \) are the slowly changing amplitude and phase of \( x_0 \). We calculate that
\[ \partial_{\tau} x_0 = \partial_{\tau} [r(T) \cos(\tau + \varphi(T))] = -r(T) \sin(\tau + \varphi(T)) \]
and
\[ \partial_{\tau \tau} x_0 = \partial_{\tau} (\partial_{\tau} x_0) = \partial_{\tau} (-r(T) \sin(\tau + \varphi(T))) = -r' \sin(\tau + \varphi) - r \cos(\tau + \varphi) \varphi', \]
where \( ' \) denotes \( \partial/\partial T \). Thus the \( \epsilon^1 \) equation is
\[ \partial_{\tau \tau} x_1 + x_1 = 2[r' \sin(\tau + \varphi) + r \varphi' \cos(\tau + \varphi)] + r \sin(\tau + \varphi)[r^2 \cos^2(\tau + \varphi) - 1]. \]
we use the identity $\sin(\tau + \varphi) \cos^2(\tau + \varphi) = 1/4[\sin(\tau + \varphi) + \sin 3(\tau + \varphi)]$ to separate the right hand side into harmonics. The result is

$$\partial_{\tau x} x_1 + x_1 = -[-2r' + r - \frac{1}{4}r^3] \sin(\tau + \varphi) + 2r\varphi' \cos(\tau + \varphi) + \frac{1}{4}r^3(\tau + \varphi).$$

We next set the coefficients of the resonant forcing terms $\sin(\tau + \varphi)$ and $\cos(\tau + \varphi)$ to zero to avoid secular terms in the solution. The resulting equations are:

$$-2r' + r - \frac{1}{4}r^3 = 0, \quad -2r\varphi' = 0.$$

We can represent the first equation as a vector field on a half-line ($r \geq 0$), using the rearranged equation

$$r' = \frac{1}{8}(4r - r^3) = \frac{1}{8}r(4 - r^2).$$

From this graph we conclude that $r^* = 2$ is a stable fixed point, so as $t \to \infty$, $r(t) \to r^* = 2$. The second equation tells us that

$$\varphi' = 0 \Rightarrow \varphi(T) = \varphi_0 = \text{constant}.$$

Thus

$$x_0(\tau, T) = 2 \cos(\tau + \varphi_0) \Rightarrow x(t) = x_0(\tau, T) + \epsilon x_1(\tau, T) + \mathcal{O}(\epsilon^2) = 2 \cos(\tau + \varphi_0) + \mathcal{O}(\epsilon).$$

The frequency is

$$\omega = \frac{d}{dt}(t + \varphi(T)) = 1 + \frac{d\varphi}{dt} = 1 + \frac{d\varphi}{dT} \frac{dT}{dt} = 1 + \epsilon\varphi' = 1 + \mathcal{O}(\epsilon^2).$$

**Bifurcations**

A bifurcation is a change in the topological structure of the phase portrait of the system when some parameter in the system changes. The types of bifurcations that we studied in one dimension are saddle-node, transcritical, and pitchfork bifurcations. These types of bifurcations behave similarly in two dimensions in that one dimensional fixed points slide toward each other along the unstable manifold. The normal form for the saddle-node bifurcation in two dimensions is

$$\begin{cases}
    \dot{x} = \mu - x^2 \\
    \dot{y} = -y
\end{cases}.$$
We have a one dimensional saddle-node bifurcation in the $x$ direction and exponential decay in the $y$ direction. Recall that the one-dimensional bifurcation diagram for the saddle-node bifurcation was

Using this information, we construct the following phase portraits for differing values of $\mu$.

In the $\mu < 0$ case, if $\mu$ is still close to zero, we have a bottleneck situation. The time spent by the phase particle in the bottleneck is given by the square root scaling law:

$$t \sim \frac{1}{\sqrt{\mu_c - \mu}}.$$ 

In this case, $\mu_c = 0$.

Dynamical Systems 10-31-05

Genetic Control Model

Suppose that there is a gene that is stimulated by the concentration of its own product protein, whose concentration will be denoted by $x$, and by the RNA copy concentration, whose concentration will be denoted by $y$. We also have parameters $a$ and $b$ that describe the degradation of the variables $x$ and $y$. This situation is called an autocatalytic feedback process. The system is given by

$$\begin{cases}
\dot{x} = -ax + y \\
\dot{y} = \frac{x^2}{1 + x^2} - by
\end{cases}.$$ 

We will show that if $a < a_c$, then the system has three fixed points, and when $a = a_c$, then two of the fixed points coalesce in a saddle node bifurcation. The nullclines are given by

$$\dot{x} = 0 \Rightarrow -ax + y = 0 \Rightarrow y = ax,$$
and
\[ \dot{y} = 0 \Rightarrow \frac{x^2}{1 + x^2} = by \Rightarrow y = \frac{x^2}{b(1 + x^2)}. \]

Also, we note that
\[ \dot{x} = -ax + y > 0 \Rightarrow y > ax, \]
and
\[ \dot{y} = \frac{x^2}{1 + x^2} - by > 0 \Rightarrow y < \frac{x^2}{b(1 + x^2)}. \]

This information gives us the graphs of the nullclines for different values of \( a \):

Obviously, we have a fixed point at \( x^* = 0, y^* = 0 \), and if \( x \neq 0 \), then
\[ ax = \frac{x^2}{b(1 + x^2)} \Rightarrow a = \frac{x}{b(1 + x^2)} \Rightarrow ab(1 + x^2) - x = 0 \Rightarrow x_{1,2} = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}. \]

From this calculation we note that at a fixed point \( x = x^* \), we have
\[ ab = \frac{x^*}{1 + x^*}, \]
which we shall use later. Now, if \( D = 1 - 4a^2b^2 > 0 \), then there are two solutions \( x_1 \) and \( x_2 \), with \( x_1 \neq x_2 \). We also have that
\[ D \geq 0 \Rightarrow 1 \geq 4a^2b^2 \Rightarrow 2ab \leq 1 \Rightarrow a \leq \frac{1}{2b} \Rightarrow a_c = \frac{1}{2b}. \]

At the bifurcation point, we have
\[ y^* = ax^* = \frac{x^*}{2b}, \]
and
\[ y^* = \frac{x^*}{b(1 + x^2)}. \]

Setting these two equal to one another, we obtain
\[ \frac{x^*}{b(1 + x^2)} = \frac{x^*}{2b} \Rightarrow 2x^* = 1 + x^* \Rightarrow x^* = 1, y^* = \frac{1}{2b}. \]

The Jacobian matrix for the system is
\[ J = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1 + x^2)} & -b \end{pmatrix}. \]
with \( \tau = -(a+b) < 0 \) and \( \Delta = ab - (2x)/(1+x^2)^2 \). At \( (0,0) \), we have \( \Delta = ab > 0 \) and \( \tau^2 - 4\Delta = (a + b)^2 - 4ab = (a - b)^2 > 0 \). Thus we have real eigenvalues whose sum is negative and have the same sign, so both must be negative. Thus we have a stable node at the origin. Now, if \( x^* < 1 \), then

\[
\Delta = ab - \frac{2x}{(1+x^2)^2} = ab - \frac{2}{1+x^2} \frac{x}{1+x^2} = ab - \frac{2ab}{1+x^2} = ab \left(1 - \frac{2}{1+x^2}\right) = ab \left(\frac{x^2-1}{1+x^2}\right) < 0.
\]

Thus, since the determinant is negative, we have a saddle point. If \( x^* > 1 \), then \( \Delta > 0 \) and \( \tau < 0 \), so both eigenvalues are negative, giving us a stable node. (To be complete, we should check that these are actually real eigenvalues.) This gives us a graph of the phase portrait:

The stable manifold separates the plain into two regions or basins of attraction for the stable nodes. The stable manifold is a threshold, in that the stable origin corresponds to the situation where the gene is silent (does not act), and a fixed point \( (x^*, y^*) \) with \( x^* > 1 \) corresponds to the action of the gene at a high level of protein concentration. When \( a = a_c \), we have a saddle-node bifurcation. We can think of the stable manifold as the \( y \)-direction and the unstable manifold as the \( x \)-direction in the graphs drawn in the last installment.

**Dynamical Systems 11-2-05**

**Zero Eigenvalue Bifurcations**

The following is a list of the types of zero eigenvalue bifurcations and their normal forms:

<table>
<thead>
<tr>
<th>Saddle-Node</th>
<th>Transcritical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x} = \mu - x^2 )</td>
<td>( \dot{x} = \mu x - x^2 )</td>
</tr>
<tr>
<td>( \dot{y} = -y )</td>
<td>( \dot{y} = -y )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Supercritical Pitchfork</th>
<th>Subcritical Pitchfork</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x} = \mu x - x^3 )</td>
<td>( \dot{x} = \mu x + x^3 )</td>
</tr>
<tr>
<td>( \dot{y} = -y )</td>
<td>( \dot{y} = -y )</td>
</tr>
</tbody>
</table>

The corresponding bifurcation diagrams for the \( x \)-directions are:
As an example, let us study the supercritical pitchfork bifurcation

\[
\begin{align*}
\dot{x} &= \mu x - x^3 \\
\dot{y} &= -y 
\end{align*}
\]

As \( \mu \) increases, we find that the phase plane changes as follows:

Notice that when \( \mu > 0 \), the origin loses its stability and two new fixed points appear at the points \( x^* = \pm \sqrt{\mu} \). When \( \mu = 0 \) or is very small and negative, there is critical slowing near the origin. When \( \mu > 0 \), we can evaluate the Jacobian at the fixed points, and we find

\[
J(x, y) = \begin{pmatrix}
\mu - 3x^2 & 0 \\
0 & -1
\end{pmatrix}, \quad J(0, 0) = \begin{pmatrix}
\mu & 0 \\
0 & -1
\end{pmatrix}, \quad J(\pm \sqrt{\mu}, 0) = \begin{pmatrix}
-2\mu & 0 \\
0 & -1
\end{pmatrix}.
\]

Note that at the bifurcation value, when \( \mu = 0 \), one of the eigenvalues at the fixed points becomes zero.

**Hopf Bifurcation**

In a bifurcation, as the parameter \( \mu \) changes, the eigenvalues of the Jacobian matrix change their positions in the complex plane. For a fixed point to be stable, the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) must be in the left half plane, i.e., \( \Re \lambda_1, \lambda_2 < 0 \). Since \( \lambda_1, \lambda_2 \) are the solutions of a quadratic equation with real coefficients, there are two possibilities:

1. \( \lambda_1, \lambda_2 \) are both real and negative.
2. \( \lambda_1, \lambda_2 \) are complex conjugates.

To destabilize the fixed point, one (or both) of the eigenvalues should move to the right half plane. In the transcritical, saddle-node, and pitchfork type bifurcations, \( \Re \lambda \) passes through 0 for one of the eigenvalues. In the Hopf bifurcation, two complex conjugate eigenvalues simultaneously move across the imaginary axis and into the right half plane.

**Example** : Consider the system

\[
\begin{align*}
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= \omega + br^2
\end{align*}
\]

Here \( \mu \) controls that stability of the fixed point at the origin, \( \omega \) gives the frequency of small-amplitude oscillations, and \( b \) determines the dependence of
frequency on amplitude. If we introduce Cartesian coordinates, the equation becomes

\[ x = r \cos \theta \Rightarrow \dot{x} = \dot{r} - r \dot{\theta} \sin \theta = (\mu r - r^3) \cos \theta - r(\omega + br^2) \sin \theta \]

\[ = x(\mu - r^2) - (\omega + br^2)y = \mu x - \omega y + O(x^2 + y^2), \]

\[ y = r \sin \theta \Rightarrow \dot{y} = r \dot{\theta} \cos \theta + \dot{r} \sin \theta = x(\mu + \omega) + (\mu r - r^3) \sin \theta \]

\[ = x(\omega + br^2) + \mu y - yr^2 = x\omega + bxr^2 + \mu y - yr^2 = \omega x + \mu y + O(x^2 + y^2). \]

Dropping the higher order terms, the system is

\[ \begin{cases} \dot{x} = \mu - \omega y \\ \dot{y} = \omega x + \mu y \end{cases} \]

and the Jacobian is

\[ J(0, 0) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \Rightarrow \lambda_{1,2} = \mu \pm i \omega. \]

As \( \mu \) increases from \( \mu < 0 \) to \( \mu > 0 \), the eigenvalues cross the imaginary axis from left to right. Looking back at the polar coordinate version of the system, we see that there is a stable fixed point for \( r \) at \( r^* = \sqrt{\mu} \). This implies that we have a stable limit cycle when \( \mu > 0 \).

Dynamical Systems 11-4-05

Subcritical Hopf Bifurcation

This type of bifurcations are potentially dangerous in engineering applications, because after the bifurcation, the solution jumps to a distant attractor, like a fixed point, limit cycle or chaotic attractor.

Example: Consider the system

\[ \begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \]

There are two attractors, the fixed point at the origin, and a stable limit at

\[ r = \sqrt{\frac{1 + \sqrt{1 - 4\mu}}{2}}. \]
At $\mu = 0$ there is a subcritical Hopf bifurcation. The unstable limit cycle separates the basins of attraction of the two attractors. As $\mu \to 0$, the unstable limit cycle shrinks to zero and overlaps the stable fixed point there.

For $\mu \geq 0$, there is only one stable limit cycle, and near the origin, solutions jump to large amplitude oscillations.

Again we see hysteresis present in this bifurcation, in that large scale oscillations cannot be "turned off" by bringing $\mu$ back to zero.

**Oscillating Chemical Reactions**

This reaction is called the Belousov-Zhabotinsky (BZ) reaction, and it concerns the chemicals Chlorine dioxide and Iodine. The variables $x$ and $y$ will represent the concentrations of $I^-$ and $ClO_2^-$, respectively. The system is

$$
\begin{cases}
\dot{x} = a - x - \frac{4xy}{1 + x^2} \\
\dot{y} = bx \left(1 - \frac{y}{1 + x^2}\right)
\end{cases}
$$

Here $a$ and $b$ are parameters depending on slow reactants also involved in the reaction, and we will assume that they are constant. We will show that the system has a closed orbit in the first quadrant for some region of parameters $\alpha$ and $\beta$. The nullclines are

$$
\dot{x} = 0 \Rightarrow (a - x)(1 + x^2) - 4xy \Rightarrow y = \frac{(a - x)(1 + x^2)}{4x},
$$

and

$$
\dot{y} = 0 \Rightarrow bx = 0 \text{ or } y = 1 + x^2.
$$

Also, we note that

$$
\dot{x} > 0 \Rightarrow y < \frac{(a - x)(1 + x^2)}{4x},
$$

$$
\dot{y} > 0 \Rightarrow 1 - \frac{y}{1 + x^2} > 0 \Rightarrow 1 + x^2 > y.
$$
We use that information to construct the following trapping region.

The fixed point is at

\[ x^* = \frac{a}{5}, \quad y^* = 1 + \left(\frac{a}{5}\right)^2, \]

since

\[ y = \frac{(a - x)(1 + x^2)}{4x} = \frac{(a - x)y}{4x} \Rightarrow 4x = a - x \Rightarrow 5x = a \Rightarrow x = \frac{a}{5}. \]

Next, we must be sure that the fixed point is a repeller. We have that

\[ J = \left( \begin{array}{cc} 3x^2 - 5 & -4x \\ 2bx^2 & -bx \end{array} \right) \frac{1}{1 + x^2} \bigg|_{x = \frac{a}{5}} \Rightarrow \Delta = \frac{5bx - 3bx^2 + 8bx^3}{(1 + x^2)^2} = \frac{5bx(1 + x^2)}{(1 + x^2)^2} > 0. \]

Since \( \Delta > 0 \), the fixed point cannot be a saddle. To ensure that the fixed point to be repelling, we want that \( \tau = \lambda_1 + \lambda_2 > 0 \). We have that

\[ \tau = \frac{3x^2 - bx + 5}{1 + x^2} \Rightarrow 3x^2 - 5 > bx \Rightarrow b < \frac{3x^2 - 5}{x} \Rightarrow b < b_c = 3x - 5 \Rightarrow x = \frac{3a - 25}{a}, \]

since \( x^* = a/5 \). Thus, when \( b < b_c \), we have an unstable spiral. A Hopf bifurcation occurs when \( b - b_c \).

Again, the fixed point loses its stability and jumps to the stable limit cycle as the bifurcation occurs.

Dynamical Systems 11-7-05
Global Bifurcations of Cycles

These bifurcations involve large regions of the phase plane. An example is a saddle-node bifurcation of two cycles where two limit cycles coalesce and annihilate. Recall that in the example of the system

\[
\begin{align*}
\dot{r} &= \mu r + r^3 - r^5 \\
\dot{\theta} &= \omega + br^2
\end{align*}
\]

there was a Hopf bifurcation at \(\mu_c^{(1)} = 0\). Here we will find \(\mu_c < 0\) for a bifurcation of limit cycles. The following phase portraits show the different behaviors of the system as \(\mu\) varies.

For \(\mu < 0\), the origin is stable, and a half-stable limit cycle is born out of the blue sky. As \(\mu\) increases, the half-stable limit cycle splits into two, one stable and one unstable. This is the bifurcation diagram.

Homoclinic Bifurcation

A limit cycle close to a saddle becomes a homoclinic orbit in this bifurcation.

Coupled Oscillations and 2D Flows on the Torus
Recall the example of the fireflies flashing which demonstrated phase locking to the stimulus. The system was given by

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + k_1 \sin(\theta_2 - \theta_1) \\
\dot{\theta}_2 &= \omega_2 + k_2 \sin(\theta_1 - \theta_2)
\end{align*}
\]

where \(\theta_1\) and \(\theta_2\) are the phases, \(\omega_1\) and \(\omega_2\) are positive and are the natural frequencies, and \(k_1\) and \(k_2\) are the coupling constants. Our coordinate system will be the torus:

or equivalently, a square with periodic boundary conditions.

First assume that \(k_1 = k_2 = 0\). Then the system is

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2
\end{align*}
\]

The trajectories are straight lines with constant slope given by

\[
\frac{d\theta_2}{d\theta_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\omega_2}{\omega_1}.
\]

There are several cases. In case (1), \(\omega_1/\omega_2 = p/q\), a rational number. Then we have closed orbits. We complete \(p\) revolutions in \(\theta_1\), while we complete \(q\) revolutions in \(\theta_2\).

This graph is a trefoil knot on the torus, since \(\theta_2\) makes \(2/3\) of a revolution while \(\theta_1\) makes one revolution.
Generally, we will have $p: q$ torus knots.

In case (2), the ratio $\omega_1/\omega_2$ is irrational. In this case, the flow is called quasi-periodic. Every trajectory winds around the torus and is never closed, and the trajectory is dense on the torus.

Now consider the case where $k_1, k_2 > 0$. We let $\varphi = \theta_1 - \theta_2$ so that $\dot{\varphi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2 - (k_1 + k_2) \sin \varphi$. Again we have several cases. Case (1) is where $|\omega_1 - \omega_2| < k_1 + k_2$. Here we have two fixed points.

In case (2), we have $|\omega_1 - \omega_2| = k_1 + k_2$. Here is where a saddle-node bifurcation occurs.

Case (3) is when $|\omega_1 - \omega_2| > k_1 + k_2$. Here there are no fixed points. Setting $\dot{\varphi} = 0$, we have that

$$\omega_1 - \omega_2 = (k_1 + k_2) \sin \varphi \iff \sin \varphi = \frac{\omega_1 - \omega_2}{k_1 + k_2}.$$ 

The phase locked solution has a constant phase difference $\varphi^*$, where

$$\varphi^* = \dot{\theta}_1 = \dot{\theta}_2 = \omega_2 + k_2 \sin \varphi = \omega_2 + \frac{k_2(\omega_1 - \omega_2)}{k_1 + k_2} = \frac{k_1 \omega_2 + k_2 \omega_1 - k_2 \omega_2}{k_1 + k_2} = \frac{k_1 \omega_2 + k_2 \omega_1}{k_1 + k_2}.$$ 

This is called the compromised frequency.
Degenerate Hopf Bifurcation

Consider the equation
\[ \ddot{x} + \mu \dot{x} + \sin x = 0. \]
As we change \( \mu \) from positive to negative, the fixed point \( x^* = 0 \) changes from a stable spiral to an unstable spiral. At \( \mu_c = 0 \), there are no limit cycles, but we have a continuous band of closed orbits, so \( x^* \) is a center. This is typical behavior for non-conservative systems to become conservative at the bifurcation point.

Poincare Maps

Consider the \( n \)-dimensional system \( \dot{x} = \vec{f}(x) \), and let \( S \) be a \( n-1 \) dimensional section of surface transverse to the flow, so that the trajectories are not parallel to \( S \). The Poincaré map \( P : S \rightarrow S \) follows the trajectory from one intersection with \( S \) to the next intersection, so that
\[ x_{k+1} = P(x_k). \]

If there exists \( x^* \) that is a fixed point for \( P(x) \), then starting at \( x^* \), the trajectory returns to \( x^* \) after some time, so there is a closed orbit in the original system. The Poincaré map is a dynamical system in discrete time.

Example : Consider the system
\[
\begin{align*}
\dot{r} &= r(1 - r^2) \\
\dot{\theta} &= 1
\end{align*}
\]
Here we take \( S \) to be the positive \( x \)-axis, i.e. \( x \geq 0 \). Then assuming that \( r(0) = r_0 \), we have that
\[ \dot{r} = r(1 - r^2), \quad r(0) = r_0 \Rightarrow r_1 = P(r_0) = r(t + 2\pi). \]
In other words, the first return to \( S \) is after a flight time of \( t = 2\pi \).

We have that
\[ \frac{dr}{dt} = \dot{r} = r(1 - r^2) \Rightarrow \int \frac{r}{r^2(1 - r^2)} \, dr = \int dt. \]
Let \( u = r^2 \), so that \( du = 2r \, dr \). Then

\[
\int \frac{r^2}{r^2(1-r^2)} \, dr = \frac{1}{2} \int \frac{du}{u(1-u)} = t + C_1.
\]

Rearranging, this becomes

\[
\int \frac{du}{u(1-u)} = \int \left( \frac{1}{u} - \frac{1}{u-1} \right) \, du = \ln \frac{u}{u-1} = 2t + C_2.
\]

We need to solve for \( u \) in this expression. To this end, note that

\[
\frac{u}{u-1} = C_3 e^{2t} \Rightarrow \frac{u-1}{u} = C_4 e^{-2t} \Rightarrow 1 - \frac{1}{u} = C_4 e^{-2t}
\]

\[
\Rightarrow \frac{1}{u} = 1 + C_5 e^{-2t} \Rightarrow u = \frac{1}{1 + C_5 e^{-2t}}.
\]

This implies that \( r = [1 + C_5 e^{-2t}]^{-\frac{1}{2}} \), and \( r = r_0 \) at \( t = 0 \) gives us that \( r_0 = (1 + C_5)^{-\frac{1}{2}} \), so that \( C_5 = r_0^{-2} - 1 \). We find that

\[
r_1 = [1 + e^{-4\pi}(r_0^{-2} - 1)]^{-\frac{1}{2}} \Rightarrow P(r) = [1 + e^{-4\pi}(r^{-2} - 1)]^{-\frac{1}{2}}.
\]

We can then use the corresponding recurrence relation to make a cobweb graph.

We see that there is a stable fixed point at \( r^* = 1 \), which corresponds to a stable limit cycle. In \( \theta \), we have rotations at unit angular velocity, and in \( r \), trajectories approach the limit cycle at \( r = 1 \).

### Linear Stability of Periodic Orbits

Suppose that \( \bar{x}^* \) is a fixed point of \( P \) corresponding to a periodic orbit. Then \( \bar{x}^* = P(\bar{x}^*) \). If we introduce a perturbation \( \bar{v}_0 \), then \( P(\bar{x}^* + \bar{v}_0) = \bar{x}^* + \bar{v}_1 \) for some \( \bar{v}_1 \) after the first return to \( S \). Expanding in Taylor series, we have

\[
P(\bar{x}^*) + \left[ \frac{\partial P}{\partial \bar{x}}(\bar{x}^*) \bar{v}_0 \right] + O(\|v_0\|^2) = \bar{x}^* + \bar{v}_1.
\]
Dropping higher order terms, we have that
\[ \vec{v}_1 = \frac{\partial P}{\partial \vec{x}}(\vec{x}^*) \vec{v}_0 = A\vec{v}_0, \]
where \( A \in \mathbb{R}^{(n-1) \times (n-1)} \) is called the linearized Poincaré map. Suppose that \( A \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \), with corresponding eigenvectors \( \vec{e}_j \). If all the eigenvalues satisfy \( |\lambda_j| < 1 \), then \( \vec{x}^* \) is linearly stable, and the corresponding periodic orbit is linearly stable. Suppose that
\[ \vec{v}_0 = \sum_{j=1}^{n-1} v_j \vec{e}_j. \]

From the fact that \( P(\vec{x}^* + \vec{v}_{k-1}) = \vec{x}^* + \vec{v}_k \), we have that
\[ P(\vec{x}^*) + \frac{\partial P}{\partial \vec{x}}(\vec{x}^*) \vec{v}_{k-1} + O(||v_k||^2) = \vec{x}^* + \vec{v}_k. \]

Dropping higher order terms, this gives us that
\[ \vec{v}_k = A\vec{v}_{k-1} = A^k \vec{v}_0 = \sum_{j=1}^{n-1} v_j \lambda_j^k \vec{e}_j. \]
If all of the eigenvalues have modulus less than one, then \( \vec{v}_k \to 0 \) as \( k \to \infty \).

The numbers \( \lambda_j \) are called characteristic or Floquet multipliers.

**Lorenz System**

The Lorenz system is given by
\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{align*}
\]

The phase space is \( \mathbb{R}^3 \), and \( \sigma, r, \) and \( b \) are positive parameters. The system was introduced by Edward Lorenz in 1963. The model consists in a fluid layer of different temperatures arranged in separate layers as in the following diagram:

This model was experimentally studied by Bernard in 1900, and the model was theoretically explained by Rayleigh in 1916, and for this reason it is called the...
R-B cell. When $\delta T = T_w - T_c$ is small enough, warm fluid, which is less dense, should rise. Fluid viscosity does not allow it to rise rapidly. When the fluid is at rest, temperature changes linearly with height.

When $\delta T$ is larger than a critical value, then convection develops, and less dense hot fluid rises up and cold fluid sinks. This creates convective rolls. The motion of the fluid and the velocity $\vec{v} = v(\vec{x}, t) = \vec{v}(\vec{x})$ don’t depend on $t$.

When $\delta T$ is greater than another critical value, then the currents start to vary in time, and the motion becomes chaotic. In the language of fluid mechanics, $r$ is the Rayleigh number, and $\sigma$ is the Prandtl number. Several important characteristics of the system are:

1. Nonlinearity
2. Symmetry: If $(x, y) \mapsto (-x, -y)$, the system doesn’t change. Thus, solutions are either symmetric or each solution has a symmetric partner.
3. Dissipation: The system exhibits volume contraction.

**Phase Volume Contraction**

Suppose that $S$ is a closed surface in phase space. In time $dt$, a patch of area $A$ on $S$ will sweep out a volume $dV = dA(\vec{f} \cdot \vec{n}) dt$, where $\vec{n}$ is the normal to $S$.

We then have that

$$V(t + dt) = V(t) + \int_S (\vec{f} \cdot \vec{n}) \, dt \, dA.$$
Thus
\[
\frac{V(t + dt) - V(t)}{dt} = \int_S (\vec{f} \cdot \vec{n}) \, dA,
\]
and as \(dt \to 0\), this becomes
\[
\dot{V} = \int_S (\vec{f} \cdot \vec{n}) \, dA.
\]
We now apply the divergence theorem to find that
\[
\dot{V} = \int_V (\nabla \cdot \vec{f}) \, dV = \int_S (-\sigma - 1 - b) \, dV = -(\sigma + 1 + b)V.
\]
Solving the following differential equation, we find that
\[
V = V_0 e^{-(\sigma + 1 + b)t}.
\]
Thus, the volume shrinks exponentially, so if all fixed points (or closed orbits if they exist) must be saddles or stable. There can be no repellers. The fixed points are given by
\[
y = x, \quad (r - 1)x - xz = 0, \quad x^2 - bz = 0.
\]
This translates to either
\[
x = y = z = 0 \quad \text{or} \quad \begin{cases} z = r - 1 \\ x^2 = b(r - 1) \end{cases}.
\]
We also have that
\[
s^2 = b(r - 1) \Rightarrow b(r - 1) \geq 0,
\]
so if \(r < 1\), there is only one fixed point at \((0, 0, 0)\). If \(r > 1\), there are three fixed points:
\[
(0, 0, 0), \quad (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, r - 1) = c^+,
\]
\[
(-\sqrt{b(r - 1)}, -\sqrt{b(r - 1)}, r - 1) = c^-.
\]
When \(r < 1\), the fixed point \((0, 0, 0)\) represents the regime for the fluid layer when there is no convection (the fluid is at rest). When \(r > 1\), \(c^+\) and \(c^-\) represent steady rolls. The fluid does not change in time, and \(c^+\) and \(c^-\) correspond to left and right turning convective rolls. At \(r = 1\), a pitchfork bifurcation occurs.
Let us now calculate the stability of the fixed point \((0,0,0)\). The Jacobian is given by
\[
J = \begin{bmatrix}
-\sigma & \sigma & 0 \\
r & -1 & 0 \\
0 & 0 & -b
\end{bmatrix},
\]
so the eigenvalues satisfy
\[
0 = \det(J - \lambda I) = \begin{vmatrix}
-\sigma - \lambda & \sigma & 0 \\
r & -1 - \lambda & 0 \\
0 & 0 & -b - \lambda
\end{vmatrix} \Rightarrow (\lambda+b)[(\lambda+1)(\lambda+\sigma)-r\sigma] = 0,
\]
so
\[
\lambda_1 = b, \quad \lambda_{2,3} = -(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}.
\]
If \(r < 1\), then \(\lambda_{2,3} < 0\). If \(r > 1\), then \(\lambda_2 < 0\) and \(\lambda_3 > 0\), so for \(r < 1\), the origin is a stable node, and for \(r > 1\), we have a saddle type fixed point. There are two incoming eigendirections and one outgoing. In the case \(r < 1\), we have global stability of the origin. We may verify this by means of a Liapunov function
\[
V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2.
\]
To see that this is indeed a Liapunov function, we must verify that it is decreasing along trajectories of the system. We show that if \((x, y, z) \neq (0, 0, 0)\), then \(\dot{V} < 0\) along the trajectory.
\[
\dot{V} = \frac{d}{dt}V = \frac{2}{\sigma}x\dot{x} + 2y\dot{y} + 2z\dot{z} = \frac{2}{\sigma}x\sigma(y-x) + 2y(rx-y-xz) + 2z(xy-bz)
\]
\[
= 2[xy - x^2 + rxy - y^2 - bz^2] = -2[x^2 + y^2 + bz^2 - (r+1)xy]
\]
\[
= -2 \left[ \left( x - \frac{r+1}{2} \right)^2 + \left( y - \left( \frac{r+1}{2} \right)^2 \right)^2 \right] y^2 + bz^2
\]
This expression is less than or equal to zero if \(r < 1\), and \(\dot{V} = 0\) only if \(x = y = z = 0\). Thus the origin is globally stable, and every trajectory approaches the origin as \(t \to \infty\).

In the case \(r > 1\), let us study the stability of \(c^+\) and \(c^-\). The Jacobian in this case is given by
\[
J = \begin{bmatrix}
-\sigma & \sigma & 0 \\
(r - z) & -1 & -x \\
y & x & -b
\end{bmatrix}(\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1).
\]
One can verify that if
\[
1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1},
\]
then \(\Re \lambda_{1,2,3} < 0\). If \(r = r_H\), then \(\lambda_{1,2} = \pm i \omega \) for some \(\omega\). If \(r > r_H\), then \(\Re \lambda_{1,2} > 0\), but \(\Re \lambda_3 < 0\). Thus we have a subcritical Hopf bifurcation.
There are no stable objects for $r > r_H$. Trajectories should approach some object of zero volume. One can show that all trajectories stay within a large ellipsoid. Trajectories are repelled from one unstable object to another unstable object. Also, flows cannot be quasi-periodic, for such flows preserve volume. Recall that last time we found that

$$V = V_0 e^{-(\sigma+b+1)t}.$$ 

Consider the example where $\sigma = 10$, $b = 8$, and $r_H = 24.74$, and suppose that $r = 28 > r_H$. The graph of $y(t)$ with initial conditions $(0, 1, 0)$ looks something like

There are random oscillations that never repeat. This is called aperiodic motion. The graphs of $x(t)$ and $z(t)$ are similar. In the $xz$-plane, this gives us the following butterfly pattern.

The trajectory switches between left and right spirals unpredictably. The limiting set is an attracting set of zero volume, called a strange attractor. Here we have strong volume contraction. After one rotation (one unit of time),

$$V = V_0 e^{-\frac{41}{3}} \approx V_0 \cdot 10^{-6}.$$ 

The fractal dimension of the limiting set is $d = 2.06$. If $r < 1$, the graphs of $x, y,$ and $z$ look like
Lorenz Map
Consider the graph of the function \( z(t) \) from the Lorenz system.

If we let \( z_k \) be a local maximum of \( z(t) \), then we can plot a graph of \( z_{k+1} \) versus \( z_k \).

The plot corresponds to a Poincaré section by the surface
\[
z = \frac{xy}{b},
\]
resulting from setting \( \dot{z} = 0 \). We will now show that the Lorenz attractor is not a stable limit cycle, so the trajectories do not settle down to a periodic motion as \( t \to \infty \). At the fixed point \( z^* \) of the graph of \( f(z) \), we have that \( |f(z^*)| > 1 \). Define \( z_n = z^* + \eta_n \). Then
\[
\eta_{n+1} = z_{n+1} - z^* = f(z_n) - f(z^*) \Rightarrow |\eta_{n+1}| = |f'(x)||z_n - z^*| > |z_n - z^*| = |\eta_n|,
\]
where the mean value theorem was used to obtain this result. This implies that there are no stable periodic orbits. Actually, we can say that there are no periodic orbits at all. Suppose that there exists a periodic solution \( z(t) \). Then the sequence \( \{z_k\} \) should be a periodic sequence as well, i.e., for some \( p \), we should have that \( z_{k+p} = z_k \) for every \( k \). Consider \( z(t) + \eta(t) \), where \( \eta(t) \) is a perturbation. Then we can generate a sequence
\[
z_0 + \eta_0, z_1 + \eta_1, \ldots, z_k + \eta_k,
\]
where \( z_{k+1} + \eta_{k+1} = f(z_k + \eta_k) \). Linearizing, we have that
\[
z_{k+1} + \eta_{k+1} = f(z_k) + \eta_k f'(z_k) = z_{k+1} + \eta_k f'(z_k)
\]
\[
\Rightarrow \eta_{k+1} = \eta_k f'(z_k) = f'(z_k)[f'(z_{k-1})\eta_{k-1}] = \cdots = f'(x_k)f'(z_{k-1})\cdots f'(z_0)\eta_0.
\]
Since we are assuming a periodic sequence, we have that
\[
\eta_{k+p} = \prod_{j=0}^{p-1} f'(z_{k+j})\eta_k \Rightarrow |\eta_{k+p}| = \prod_{j=0}^{p-1} |f'(z_{k+j})||\eta_k| > |\eta_k|,
\]
since \( |f'(z_{k+j})| > 1 \) for \( j = 0, 1, \ldots, p - 1 \). We now know the following things about the system:
1. There is no attracting fixed point at \( \infty \).
2. There are no attracting fixed points in \( \mathbb{R}^3 \).
3. There are no stable limit cycles.
4. There are no quasi-periodic orbits.

However, the attracting set \( A \) for \( \dot{x} = f(x) \) should satisfy

1. \( A \) is an invariant set: \( x(0) \in A \Rightarrow x(t) \in A \ \forall \ t \geq 0. \)
2. \( A \) attracts some open set of initial conditions: there exists \( U \supset A \) such that \( x(0) \in U \) implies \( \rho[x(t), A] \to 0 \) as \( t \to \infty \), where \( \rho \) denotes distance. The largest such set is called the basin of attraction for \( A \).
3. \( A \) is a minimal set, in the sense that there is no set \( B \) which satisfies (1) and (2) and such that \( B \subset A, B \neq A \).

**Example**: Consider the system

\[
\begin{cases}
\dot{x} = x - x^3 \\
\dot{y} = -y
\end{cases}
\]

and the set \( I = \{(x, y) : -1 \leq x \leq 1\} \). The phase portrait is

We note that

1. \( I \) is invariant.
2. \( I \) attracts an open set of initial conditions (\( I \) attracts all of \( \mathbb{R}^2 \)).
3. \( I \) is not minimal—the fixed points \((\pm 1, 0)\) and \((0, 0)\) both satisfy (1) and (2).

**Definition 6** A strange attractor is an attractor which exhibits a sensitive dependence on initial conditions.

We have that

\[ |\delta(t)| \approx |\delta_0|e^{\lambda t} \Rightarrow \ln |\delta| = \ln |\delta_0| + \lambda t. \]
For the Lorenz attractor, $\lambda = .9$ is the Liapunov exponent. This implies that neighboring trajectories separate exponentially fast.

**Definition 7** Chaos is an aperiodic long term behavior in a deterministic system with sensitive dependence on initial conditions (nearby trajectories separate exponentially fast).

**Iterated Maps**

Iterated maps are dynamical systems which are discrete in time, and have the form $x_{k+1} = f(x_k)$, $x \in \mathbb{R}^p$, where $x_0$ is given. An interesting feature of iterated maps is that chaos can occur even in one dimension.

**Definition 8** Suppose that $x^*$ is a fixed point of an iterated map, i.e., that $x^* = f(x^*)$. Then $x^*$ is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$, then $|x_k - x^*| < \epsilon$ for all $k$.

To determine stability, we linearize the map and consider a nearby orbit

$$x_n = x^* + \eta_n.$$ 

Then

$$x_{n+1} + \eta_{n+1} = f(x^* + \eta_n) = f(x^*) + \frac{\partial f}{\partial x}(x^*)\eta_n + \mathcal{O}(|\eta_n|^2).$$

Thus,

$$\eta_{n+1} = \frac{\partial f}{\partial x}(x^*)\eta_n.$$ 

If

$$A = \frac{\partial f}{\partial x}(x^*),$$

and if all the eigenvalues of $A$ satisfy $|\lambda_i| < 1$, then $x^*$ is stable. If there is a $j$ such that $|\lambda_j| > 1$, then $x^*$ is unstable.

**Example** : Consider the recurrence relation $x_{n+1} = x_n^2$. Then a fixed point $x^*$ must satisfy $x^* = x^*^2$ so either $x^* = 0$ or $x^* = 1$. We see that if

$$\lambda = f'(x^*) = 2x^*,$$

then at 0, $\lambda = 0$, so the fixed point is stable, and at $x^* = 1$, $\lambda = 2$, so the fixed point is unstable. We can also analyze this information using a cobweb diagram.
Iterated Maps

Consider the iterated map

\[ x_{n+1} = \cos x_n. \]

Solving the equation \( x = \cos x \), we find that the map has a fixed point at \( x^* \approx 0.74 \). To see that the fixed point is stable, we calculate that

\[ \lambda = f'(x^*) = -\sin(0.74) = -0.67, \]

so the slope of the tangent line at the fixed point is less than one, which means that the fixed point is stable. If \( \lambda = f'(x^*) < 0 \), then we see damped oscillations, and if \( \lambda > 0 \), then the convergence is monotonic.

Also consider the quadratic map that is the discrete time analog of the logistic equation in ODE’s. It is given by

\[ x_{n+1} = rx_n(1 - x_n), \]

where \( r \) is a parameter describing the growth rate of the population. If we let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = rx(1 - x) \), then we see that

\[ \max_{x \in [0,1]} f(x) = f\left(\frac{1}{2}\right) = \frac{r}{4}, \]

so if \( 0 \leq r \leq 4 \), then \( f : [0,1] \to [0,1] \). We have that \( x^* = 0 \) is a stable fixed point for \( r < 1 \), and by examining the equation \( x = rx(1 - x) \), we see that \( x^* = 0 \) is a fixed point for all values of \( r \). Also, \( f'(0) = r(1 - 2x)|_{x=0} = r \), so at \( r = 1 \), \( x^* = 0 \) becomes unstable. At this value of \( r \), another fixed point also appears in a transcritical bifurcation.

The location of the other fixed point is given by

\[ x = rx(1 - x), \ x \neq 0 \Rightarrow 1 = r(1 - x) \Rightarrow x^* = 1 - \frac{1}{r}. \]
As \( r \) increases, \( x^* = 1 - 1/r \) becomes unstable, since the curve becomes taller and taller and the slope of the curve at the nonzero fixed point decreases towards -1.

\[
f'(x^*) = r - 2rx|_{x=1-\frac{1}{r}} = r - 2r \left(1 - \frac{1}{r}\right) = 2 - r.
\]

In order to have a stable fixed point, we want values of \( r \) such that \(|f'(x^*)| < 1\).

We find that the range of values that will work is

\[
2 - r < 1 \Rightarrow r < 1, \quad -(2 - r) < 1 \Rightarrow r < 3 \Rightarrow 1 \leq r \leq 3.
\]

The critical slope is when \( f'(x^*) = -1 \), which is when \( r = 3 \). At this value, we have a flip bifurcation, which caused period doubling.

This 2-periodic sequence corresponds to period 2 cycles in the size of the population.

In summary, \( f : [0, 1] \rightarrow [0, 1] \) has a fixed point at \( x^* = 0 \) for all values of \( r \), and it is stable for \( r < 1 \). The fixed point at \( x^* = 1 - 1/r \) exists for \( 1 \leq r \leq 4 \), and it is stable for \( 1 < r < 3 \). When \( r = 3 \), we have a period 2 cycle. At \( r = 3.449 \), we have a period 4 cycle, and so on. For \( r > r_\infty = 3.5699 \), all the motion becomes chaotic and \( \{x_n\} \) never settles down to a fixed point or periodic orbit.

Let us investigate further the period 2 cycle that arises. If \( x_1 = f(x_0) \), \( x_0 = f(x_1) \), then \( x_0 \) is a fixed point of the second iterate map \( f^2(x) = f(f(x)) \). To find \( x_0 \), we solve

\[
x = f(f(x)),
\]

which in this case is the quartic equation

\[
x = r(rx(1-x))[1-rx(1-x)].
\]

We already know two solutions \( x = 0 \) and \( x = 1 - 1/r \), so we divide these factors off, leaving us with a quadratic equation whose solution is

\[
x_{0,1} = \frac{(r + 1) \pm \sqrt{(r - 3)(r + 1)}}{2r}.
\]
This implies that $r > 3$ for real solutions.

If $r > 1 + \sqrt{6}$, then the fixed point of $f^2(x)$ becomes unstable, and a 2 cycle of $f^2(x)$ appears. Such a 2 cycle of $f^2(x)$ corresponds to a 4 cycle of $f(x)$. In other words, we may say that $f^3(x)$ has a fixed point. To show that the 2 cycle is stable for $3 < r < 1 + \sqrt{6}$, we note that

$$\lambda = \frac{d}{dx} [f(f(x))] \bigg|_{x_0} = f'(f(x))f'(x)|_{x_0} = f'(f(x_0))f'(x_0) = f'(x_1)f'(x_0) = r(1-2x_1)r(1-2x_0) = r^2[1-2(x_0+x_1)],$$

since

$$x_0 + x_1 = \frac{r + 1}{r}, \quad x_0x_1 = \frac{(r + 1)^2 - (r - 3)(r + 1)}{4r^2} = \frac{r + 1}{4r^2} [r + 1 - r + 3].$$

We want $|\lambda| < 1$ for the 2 cycle to be stable, so we require that

$$|4 + 2r - r^2| < 1 \Rightarrow r < 1 + \sqrt{6}.$$ 

This gives us the following bifurcation diagram:

---

Consider the logistic map $x_{n+1} = f(x_n)$, where $f(x) = rx(1-x)$. Then for $0 < r < 1$, we have that $x_n \to 0$ as $n \to \infty$. For $1 < r < r_1 = 3$, we have that $x_n \to 1 - 1/r$. For $r_1 < r < r_2 = 1 + \sqrt{6}$, a bifurcation of period doubling has occurred, and $x_n$ approaches a 2 cycle. For $r_2 < r < r_3$, $x_n$ approaches a 4 cycle, etc. If we consider the sequence $\{r_n\}$, we see that $r_n \to r_\infty$, and

$$\lim_{n \to \infty} \frac{r_{n-1} - r_n}{r_n - r_\infty + 1} = \delta = 4.669 \ldots$$

This is a universal constant for all such maps $f$, which are concave down with a single maximum, called unimodal functions. Figenbaum proved that no matter what unimodal map is iterated, the number $\delta$ is universal.
Another such universal constant is
\[
\lim_{n \to \infty} \frac{d_n}{d_{n+1}} \rightarrow \alpha = -2.5029 \text{ as } n \to \infty.
\]

**Liapunov Exponent** Consider the map \( x_{n+1} = f(x_n) \) and a small perturbation \( x_{n+1} + \epsilon_{n+1} = f(x_n + \epsilon_n) \).

We have that \( \epsilon_{n+1} = f(x_n + \epsilon_n) - f(x_n) \approx \epsilon_n f'(x_n) \). Thus
\[
|\epsilon_1| = |f'(x_0)||\epsilon_0|,
|\epsilon_2| = |f'(x_1)||\epsilon_1|,
|\epsilon_n| = |f'(x_{n-1})||\epsilon_{n-1}|.
\]

This gives us that
\[
|\epsilon_n| = \prod_{k=0}^{n-1} |f'(x_k)||\epsilon_0|.
\]

If we take the logarithm of each side of the equation above, we find that
\[
\ln \left| \frac{\epsilon_n}{\epsilon_0} \right| = \sum_{k=0}^{n-1} \ln |f'(x_k)|.
\]

**Definition 9** *The Liapunov exponent of the map* \( x_{n+1} = f(x_n) \) *is defined to be*
\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|.
\]

We then have that
\[
\ln \left| \frac{\epsilon_n}{\epsilon_0} \right| \approx n\lambda
\]
for large \( n \), or equivalently,
\[
\epsilon_n \approx \epsilon_0 e^{n\lambda}.
\]

If \( \lambda > 0 \), then the \( \epsilon_n \) grow exponentially, and if \( \lambda < 0 \), then \( \epsilon_n \to 0 \).

**Example**: Consider the map \( x_{n+1} = ax_n \). Then \( x_{n+1} + \epsilon_{n+1} = a(x_n + \epsilon_n) \), so that
\[
\epsilon_{n+1} = a\epsilon_n, \text{ and } |\epsilon_n| = |a|^n |\epsilon_0| = |\epsilon_0| e^{n\lambda},
\]
where \( \lambda = \ln |a| \). Then if \( |a| < 1 \), we have a contraction mapping, and the iteration converges to a fixed point, and if \( |a| > 1 \), the perturbation grows exponentially.

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Superstable Fixed Points

Given a map \( x_{n+1} = f(x_n) \), a point \( x^* \) is a fixed point if \( f(x^*) = x^* \). The value \( \lambda = f'(x^*) \) determines the stability of \( x^* \). If \( |\lambda| < 1 \), then \( x^* \) is stable, and if \( \lambda = 0 \), then \( x^* \) is called superstable. Suppose that \( x_n = x^* + \eta_n \). Then expanding in Taylor series, we have that

\[
\eta_{n+1} = f'(x^*) \eta_n + C \eta_n^2 + O(\eta_n^3).
\]

Thus, if \( \lambda = f'(x^*) \neq 0 \), we have that \( \eta_n \approx \lambda^n \eta_0 \), but if \( \lambda = 0 \), we have \( \eta_n \approx C \eta_n^2 \). If we multiply this expression through by \( C \) and let \( \xi_n = C \eta_n \), then we have that \( \xi_{n+1} = \xi_n^2 \), so \( \xi_n = (\xi_0)^{2^n} \). This tells us that perturbations die out much faster in the superstable case.

Newton’s Method and Superstability

The roots of a differentiable function \( g(x) \) correspond to superstable fixed points of Newton’s map

\[
x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = f(x_n).
\]

If \( x^* \) is a fixed point of \( f(x) \), then

\[
f(x^*) = x^* \Rightarrow x^* = x^* - \frac{g(x^*)}{g'(x^*)}.
\]

The derivative of \( f \) evaluated at \( x^* \) is

\[
1 - \frac{(g'(x^*))^2 - g(x^*)g''(x^*)}{(g'(x^*))^2} = \frac{g(x^*)g''(x^*)}{(g'(x^*))^2}.
\]

If \( g'(x^*) \neq 0 \), \( g''(x^*) \) is bounded, and \( g(x^*) = 0 \), then

\[
f'(x^*) = \frac{g(x^*)g''(x^*)}{(g'(x^*))^2} = 0.
\]

Thus \( x^* \) is a superstable fixed point of Newton’s Method.

Logistic Map

Recall that the logistic map is given by

\[
f(x) = rx(1-x) = rx - rx^2,
\]

and that for \( 0 < r < 1 \), we have a 1-cycle, i.e., \( x_n \to 0 \). If \( 1 < r < r_1 \), then we have another 1-cycle, \( x_n \to 1 - 1/r \). If \( r_1 < r < r_2 \), then we have that \( x_n \) approaches a 2-cycle, since a bifurcation of period doubling has occurred. In the same manner, if \( r_2 < r < r_3 \), \( x_n \) approaches a 4-cycle, and so on. Actually, as we look closer at what is happening as \( r \) is increased, we see that the fixed point becomes superstable at some value of \( r \). A similar thing happens with the fixed point of the second iterate map \( (f^2(x)) \). The fixed point, corresponding
to a 2-cycle of the original system, becomes superstable, then unstable, then becomes a 2-cycle (a 4-cycle for the original system). If we look even closer, we see that the behavior of the second iterate is like the first, because the regions of the graph of the second iterate that is involved looks just like the first iterate. It is just a scaled and shifted version.

Let us find the value \( r = R_0 \) for which \( x^* = 1 - 1/r \) is superstable. We see that

\[
f'(x^*) = r(1 - 2x^*) = r \left(1 - 2 + \frac{2}{r}\right) = r \left(\frac{2}{r} - 1\right),
\]

and we seek a solution that satisfies \( 1 < r < 3 \). Thus we have that \( R_0 = 2 \). Similarly, \( R_1 \) is the value for which the sequence

\[x_0, x_1, x_0, x_1, \ldots\]

is superstable. We follow the same procedure to find \( R_1 \).

\[
\lambda = \frac{d}{dx} [f(f(x))]_{x=x_0} = f'(f(x_0))f'(x_0) = f f'(x_1)f'(x_0).
\]

We want \( \lambda = 0 \), so we must have either \( f'(x_1) = 0 \) or \( f'(x_0) = 0 \). If \( f'(x) = r(1 - 2x) = 0 \), then \( x = 1/2 \). Recall also that we had an expression for the points of the 2-cycle,

\[
x_{0,1} = \frac{(r + 1) \pm \sqrt{(r - 3)(r + 1)}}{2r} = \frac{1}{2}
\]

We solve this for \( r = R_1 \). We can repeat the procedure to obtain \( R_2 \) corresponding to a superstable 4-cycle,

\[x_0, x_1, x_2, x_3, x_0, x_1, x_2, x_3, \ldots\]

In a similar manner, we find that

\[
\frac{d}{dx} [f^4(x)]_{x=x_0} = f'(x_0)f'(x_1)f'(x_2)f'(x_3).
\]

Again we must have at least one of \( x_0, x_1, x_2, x_3 = 1/2 \). If we express the location of the fixed points as a function of \( r \), then we can determine the value of \( R_2 \). We find that similar dynamics are repeated at each stage: a \( 2^n \) period cycle is born, then becomes superstable, then loses stability and a bifurcation of period doubling occurs.
If we translate the origin and scale, we see that the reason is that the graphs of the higher iterates of $f$ look similar to the original $f$ on a small scale. This can be seen even more clearly when the proper shift and scaling is applied (scale factor $\alpha = -2.5$).

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Fractals

Fractals are geometric shapes with fine structure at an arbitrarily small level. To study these, we will need some background in set theory.

We say that sets $X$ and $Y$ have the same number of elements if there exists a one to one correspondence between the elements of the sets.

Example: The sets

$$X = \{1, 3, 5, \ldots\}$$

and

$$Y = \{2, 4, 6, \ldots\}$$

have the same number of elements as the set

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$ 

If there exists a bijection between a set $X$ and $\mathbb{N}$, then $X$ is called countable.

Example: The set of rational numbers $\mathbb{Q}$ is countable. It can be counted as outlined in the following diagram:

However, the set $\mathbb{R}$ is not countable. Even the set $(0, 1)$ is not countable. This can be seen as follows. Suppose that we can list all $y \in (0, 1)$ in decimal form as

$$y_1 = 0.y_{11}y_{12}y_{13}\cdots$$

$$y_2 = 0.y_{21}y_{22}y_{23}\cdots$$

$$\vdots$$

Then let $z = 0,y_{11}y_{22}y_{33}\cdots$, where $y_{ii}$ is chosen so that $\bar{y}_{ii} \neq y_{ii}$ for each $i$. Then $z \in (0, 1)$, but $z$ is not in our list.

Another way of saying that two sets have the same number of elements is saying that they have the same cardinality. Thus two sets $X$ and $Y$ have the same cardinality if there is an invertible map between $X$ and $Y$. The cardinality of $(0, 1)$ is called the continuum. Since we have seen that $\mathbb{Q}$ is countable, it must be the case that $\mathbb{R} - \mathbb{Q}$, the set of irrational numbers, is uncountable.
Decimal Shift Map

Consider the map \( x_{n+1} = 10x_n \pmod{1} \). Here \( \pmod{1} \) means that we take the fractional part of \( x \). For example, if \( x_n = 0.12579 \ldots \), then \( x_{n+1} = 0.2579 \ldots \). This map has countably many periodic orbits (rational numbers) and uncountably many aperiodic orbits (irrational numbers).

We can easily see that any interval \([a,b]\) has the cardinal number of the continuum, for if \( U = [0,1] \), then \( y = a + (b-a)x \) is a bijection from \( U \) to \( A \). Also, we can represent each element in \( U \) as a binary number. If \( x \in U \), then
\[
x = 0.a_1a_2a_3\ldots = \sum_{k=1}^{\infty} \frac{a_k}{2^k},
\]
where \( a_k \) can be either 0 or 1.

Cantor Set

To construct the Cantor set, we start with \([0,1]\), and we remove the middle third of this interval. Then from the two remaining intervals, we remove both of their middle thirds, and so on, removing the middle thirds of the intervals that remain at each step. The process looks something like this:

The Cantor set is what remains after infinitely many steps. To calculate the measure of the Cantor set, we sum up the lengths of the intervals if \( \bar{S} = [0,1] - C \), then
\[
\mu(\bar{S}) = \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = \frac{1}{1 - \frac{2}{3}} = 1.
\]

Thus \( \mu(C) = 0 \). We can also characterize the Cantor set as the set of all points in \([0,1]\) that do not have 1 in their base three expansion. If we expand \( x \in [0,1] \) in base three, then
\[
x = 0.a_1a_2a_3\ldots = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots,
\]
where \( a_k \) is 0, 1, or 2. If we look closely at the intervals that are removed, we see that at the \( k \)th step, we remove all numbers with 1 in the \( k \)th place after the “decimal” point in the ternary expansion.

The Cantor set is uncountable, in that it has “as many points” as \([0,1]\). We can write all points in the unit interval in base 2, and then identify the 2’s in
the base three representation of \( C \) with the 1’s in the base two representation of \([0, 1]\).

**Dimension**

Most of us are somewhat familiar with the idea of dimension. For example, we know that a line or a smooth curve is a one dimensional object, while a plane or surface has dimension two.

Consider the Von Koch curve. To construct this curve, we start with the unit interval and remove the middle third. We then replace it with two edges of an equilateral triangle, as shown below.

Repeating this process *ad infinitum*, we obtain the Von-Koch curve. At each successive iteration, the length of the curve is increased by a factor of \( 4/3 \), so when we have obtained the limit curve (after completing an infinite number of iterations), any point on the curve is infinitely far away from any other point on the curve.

**Similarity Dimension**

Suppose that we have a geometric object, and we find that if we shrink this object in size by a factor \( r \), then we can rebuild the original with a certain number of copies of the shrunken object. Then we can define a dimension \( d \) for the object by

\[
m = r^d,
\]

where \( m \) is the number of copies of the shrunken object needed to rebuild the original and \( r \) is the shrinking factor. We can rewrite this formula as

\[
d = \frac{\ln m}{\ln r}.
\]

For example, if we take a square in the plane and shrink it by a factor of 2, then we can rebuild the square with four shrunken squares. This gives us that

\[
4 = 2^d \implies d = 2.
\]
We can do the same thing for the Cantor Set. At each step, if we shrink the set that is left by a factor of 3, we can rebuild the original with two copies. Thus $d = \frac{\ln 2}{\ln 3} = .63$. Similarly, if at some stage we shrink the Von Koch curve by a factor of 3, we would require 4 shrunken curves to rebuild the original. Thus $d = \frac{\ln 4}{\ln 3} = 1.26$.

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Recall that any point in the Cantor set can be represented as a base 3 number

$$x = 0.b_1b_2b_3\ldots,$$

where $b_k \in \{0, 2\}$. Any point in $[0, 1]$ has a base 2 representation

$$y = 0.a_1a_2a_3\ldots,$$

where $a_k \in \{0, 1\}$. Thus we find a one to one map from $[0, 1]$ to the Cantor set by setting $b_i = 2a_i$, and it must be that the Cantor set has the same cardinality as the unit interval.

The Serpinski carpet follows basically the same construction as the Cantor set, but in two dimensions. The Serpinski carpet has no ones in the base three representations of its coordinates $(x, y)$.

We can easily calculate the area of the deleted set as

$$\left(\frac{1}{3}\right)^2 + 8\left(\frac{1}{3^2}\right)^2 + (8)^2\left(\frac{1}{3^4}\right)^2 + \cdots = \sum_{i=1}^{\infty} \frac{8^{i-1}}{3^{2i}} = \frac{\frac{1}{9}}{1 - \frac{9}{9}} = 1.$$

Since the deleted areas sum to 1, the area of the Serpinski carpet is zero. We can also calculate the similarity dimension of the Serpinski Carpet by shrinking it by a factor of three at some step and then using eight of the shrunken squares to build up the original. Thus

$$d = \frac{\ln m}{\ln r} = \frac{\ln 8}{\ln 3}.$$ 

In three dimensions, the same process produces an object called the Menger sponge.
If we scale by \( r = 3 \), then we can rebuild the object at the first step using \( m = 20 \) of the scaled cubes (we deleted 7). Then

\[
d = \frac{\ln 20}{\ln 3}.
\]

Another such example is the ”even fifths” Cantor set, where at each step the remaining intervals are divided into fifths and the even numbered intervals (from left to right) are thrown away.

In this case we have

\[
d = \frac{\ln 3}{\ln 5}.
\]

**Definition 10** A set \( S \) is called a topological Cantor set if (1) \( S \) is totally disconnected and (2) \( S \) contains no isolated points.

The cross section of a strange attractor is often a topological Cantor set, as is seen in the following example. Recall that the logistic map is given by \( x_{n+1} = rx_n(1 - x_n) \), and that the bifurcation values of \( r \) approached a limiting value \( r_\infty = 3.5699 \). We can build up the attractor recursively by looking at the points \( R_0, R_1, R_2, \ldots \) where the map has a superstable \( 2^n \) cycle.

As \( n \to \infty \), the sequence of points indicated in the diagram above approaches a topological Cantor set. The points are separated by gaps of various sizes, so the limiting set is not self-similar, but it is uncountable.

**Box Dimension**

Consider a collection of boxes of side length \( \epsilon \) that cover a segment of a curve in the plane. The number of boxes required is approximately

\[
N(\epsilon) \approx \frac{L}{\epsilon},
\]

where \( L \) is the actual length of the curve. If we do the same thing with a plane region of area \( A \), then we have

\[
N(\epsilon) \approx \frac{A}{\epsilon^2}.
\]
In general, we can say
\[ N(\epsilon) \sim \frac{1}{\epsilon^d} \ln N(\epsilon) = d \ln \left(\frac{1}{\epsilon}\right) \Rightarrow d = \lim_{\epsilon \to 0} \frac{\ln(N(\epsilon))}{\ln \left(\frac{1}{\epsilon}\right)}. \]

In the construction of the Cantor set, \( S_\infty \) is covered by \( S_n \) for every \( n \), so we have \( 2^n \) subintervals in \( S_n \) of length \((1/3)^n\). This gives us that
\[ d = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln 2}{\ln 3}. \]

This idea of dimension is more general than that of similarity dimension, since it can be used on non-self-similar sets. Consider as an example the random Serpinski carpet, which is constructed just as outlined above, except that at each stage a random square is deleted rather than the center square.

At each step, \( S \) is covered by \( 8^n \) squares, each of side length \( \epsilon = (1/3)^n \). Then
\[ d = \lim_{n \to \infty} \frac{\ln 8^n}{\ln 3^n} = \frac{\ln 8}{\ln 3}. \]

**Strange Attractors**

The mechanism of creating a strange attractor is repeated stretching and folding. Consider some set (a blob) of initial conditions. The flow contracts the blob in some directions (the system is usually dissipative) and stretches it in other directions. All this motion is also sensitively dependent on initial conditions. The stretched blob is folded back so that it stays in a bounded region.

We can use this sort of folding and flattening process to make what is known as Philo Pastry. It is called this because a "flaky" layered structure results, like
that of a croissant. The steps involved are to flatten and stretch a blob, then fold it upon itself, and start all over again.

When we have done this an infinite number of times, the resulting set has a cross-section that is a Cantor-like set.

Another example is the Baker’s map, \( B : [0,1]^2 \rightarrow [0,1] \), which is given by
\[
(x_{n+1}, y_{n+1}) = \begin{cases} 
(2x_n, ay_n) & \text{for } 0 \leq x_n \leq \frac{1}{2} \\
(2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \leq x_n \leq 1
\end{cases}, \quad a < \frac{1}{2}.
\]

We can think of \( B \) as the composition of two transformations: (1) The square is stretched and flattened into a \( 2 \times a \) rectangle. (2) The rectangle is cut in half and the right half is stacked on top of the left half at the level \( y = \frac{1}{2} \). For \( a < \frac{1}{2} \), the Baker’s map \( B \) has a fractal attractor \( A \) that attracts all of the orbits. Let us find this attractor. We wish to construct a set \( A \) such that for any initial conditions \((x_0, y_0)\) we have
\[
d(A, B^n(x_0, y_0)) \rightarrow 0 \text{ as } n \rightarrow \infty,
\]
where \( d \) denotes distance. We follow these steps: First we construct the attractor \( A \). Let \( S \) be the square \([0,1]^2\). This includes all possible initial conditions. At successive iterations, we have the following:

At step \( n \), we have \( 2^n \) strips of height \( a^n \). The limiting set \( A = B^\infty(S) \) is a fractal, a Cantor-like set of line segments. We know that a limiting set exists, since we have a successive sequence of nested sets
\[
B^{n+1}(S) \subset B^n(S).
\]
Each \( B^n(S) \) is a compact set, and the countable intersection of a nested family of compact sets is compact and nonempty. Thus \( A \subset B^n(s) \) for every \( n \). The set \( A \) that attracts all the orbits \( B^n(x_0, y_0) \) is in the strips in \( B^n(S) \), and so all
the points of $B^n(S)$ are within a distance $a^n$ of the attracting set, or in other words

$$\max_{x \in B^n(S)} d(x, A) \leq a^n,$$

and $a^n \to 0$ as $n \to \infty$.

We can also find the box dimension of the set $A$. We know that $A \subset B^n(S)$ for each $n$, and that $B^n(S)$ consists of $2^n$ strips of height $a^n$ and length 1. Thus $A$ is covered with squares of side $\epsilon = a^n$, so $1/\epsilon = a^{-n}$. We need $a^{-n}$ of these boxes to cover each strip. Thus $N(\epsilon) = 2^n a^{-n}$. This gives us that

$$d = \frac{\ln 2}{\ln a} = \frac{\ln 2 - \ln a}{-\ln a} = 1 - \frac{\ln 2}{\ln a}.$$

As $a \to 1/2$, $d \to 2$.

**Example**: If $a = 1/4 = (1/2)^2$, then

$$d = 1 + \frac{\ln \frac{1}{2}}{\ln \left(\frac{1}{2}\right)^2} = 1 + \frac{1}{2} = \frac{3}{2}.$$

**Dynamical Systems 12-7-05**

Recall the Baker’s map from the last installment. For $a < 1/2$, $B(S)$ shrinks areas in phase space for every region $R \subset S$, i.e., the area of $B(R)$ is less than the area of $R$. The map $B$ elongates $R$ by a factor of 2 in the $x$-direction and contracts (or flattens) by a factor $a$ in the $y$-direction. Thus

$$\text{area}(B(R)) = 2a \cdot \text{area}(R) < \text{area}(R),$$

because $a < 1/2$. This area contraction for $B$ is similar to the volume contraction that we pointed out while studying the Lorenz equations. From this we find that the attractor $A$ has area($A$)=0, and $B$ does not have any repelling fixed points. If $a = 1/2$, $B$ is area preserving, and the orbits never settle down to a lower dimensional attractor.

There are now two main types of systems, the dissipative systems which are characterized by the presence of volume contraction, and the Hamiltonian area-preserving systems. The dissipative systems often model friction or viscosity, and there is a lower dimensional attractor that attracts all orbits that start near it. The Hamiltonian/conservative systems do not have any attractors.

Another example of maps that generate strange attractors are the Smale horseshoe map:

Others are the Henon map and the Rossler system.