

Research Statement

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The purpose of this document is to sketch my current research interests, the work I have done so far, and some open problems and questions that I would like to continue pursuing. I have subdivided the presentation in two sections: the first one provides context and background for the second one. In order to make the second section more self-contained, I describe with some detail the TQFT of Admissible Covers; to avoid being repetitious, since the general set-up is extremely parallel, I was much faster in presenting the analogous theory of Bryan and Pandharipande. In no way this stylistic choice intends to take any credit away from them, who must be considered the “fathers” of this kind of theories, and whose work has been amazing inspiration and support for mine.

Background

(1+1)Topological Quantum Field Theories

A *(1+1) dimensional Topological Quantum Field Theory* (TQFT from now on) is a functor of tensor categories:

$$\mathcal{T} : \mathbf{2Cob} \longrightarrow \mathbf{Rmod}.$$

On the right hand side, we have the familiar category of modules over a ring R .

Let us now briefly describe the category $\mathbf{2Cob}$:

Objects: objects are 1 dimensional oriented closed manifolds, i.e., finite disjoint unions of oriented circles.

Morphisms: morphisms are (equivalence classes of) oriented cobordisms between two objects. We can think of them as oriented topological surfaces with oriented boundary components.

Composition: we compose two morphisms by simply concatenating them; equivalently, we glue negatively oriented boundary components of one surface to positively oriented boundary components of the other.

Tensor Structure: the tensor structure is given by disjoint union.

Since all objects in $\mathbf{2Cob}$ are generated from S^1 by tensoring, the R -module $H := \mathcal{T}(S^1)$ plays a special role, and it is called the *Hilbert Space* of the TQFT.

It is easy to show that a TQFT gives the Hilbert space H the structure of a Commutative Frobenius Algebra. A TQFT is called *semisimple* if H is a semisimple Frobenius Algebra.

Philosophically, a TQFT is a way to give structure to invariants of surfaces that “behave well” under gluing. Since any topological surface with boundary can be decomposed into discs, annuli and pairs of pants, a TQFT is uniquely determined by functoriality once it has been described on these basic building blocks.

Dijkgraaf’s TQFT of Covers.

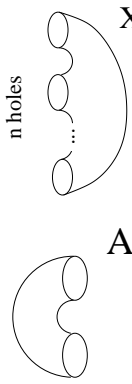
In the early 1990’s ([D]), the mathematical physicist Robbert Dijkgraaf noticed that a TQFT approach yields a beautiful and elegant solution to a classical mathematical problem: counting ramified and unramified covers of a topological surface.

Let (X, p_1, \dots, p_n) be an n -marked smooth projective curve. Let $\underline{\eta} = (\eta_1, \dots, \eta_n)$ be a vector of partitions of the integer d . We define the *Hurwitz Number*:

$$H_d^X(\underline{\eta}) = \text{“number” of } \left\{ \begin{array}{l} \text{degree } d \text{ covers} \\ C \xrightarrow{\pi} X \text{ such that :} \\ \bullet \pi \text{ is unramified over } X \setminus (p_1, \dots, p_n); \\ \bullet \pi \text{ ramifies with profile } \eta_i \text{ over } \pi_i. \end{array} \right\}$$

The above number is divided by the number of automorphisms of such covers. We define the TQFT \mathcal{D} as follows:

1. the ground ring is simply \mathbb{C} ;
2. the Hilbert space is $H = \bigoplus_{\eta+d} \mathbb{C}e_{\eta}$;
3. morphisms are assigned according to the prescription:



$$\begin{array}{l} \begin{array}{c} \text{n holes} \\ \text{X} \end{array} \xrightarrow{\mathcal{D}} \mathcal{D}(X) : \begin{array}{ccc} H^{\otimes n} & \longrightarrow & \mathbb{C} \\ e_{\eta_1} \otimes \dots \otimes e_{\eta_n} & \mapsto & H_d^X(\underline{\eta}). \end{array} \\ \begin{array}{c} \text{A} \end{array} \xrightarrow{\mathcal{D}} \mathcal{D}(A) = \sum \zeta(\eta) e_{\eta} \otimes e_{\eta}. \end{array}$$

The coefficient $\zeta(\eta)$ is just a combinatorial factor that makes things “work”.

This is indeed a well defined TQFT. Furthermore, the Hilbert space is isomorphic, as a Frobenius Algebra, to the class algebra of the symmetric group in d letters, $\mathcal{Z}(\mathbb{C}[S_d])$.

It is well known to representation theorists that $\mathcal{Z}(\mathbb{C}[S_d])$ is semisimple, with semisimple basis indexed by irreducible representations ρ of S_d . This allows Dijkgraaf to recover the classical Burnside formula, expressing the number of unramified covers of a genus g curve:

$$\sum_{\rho} \left(\frac{d!}{\dim(\rho)} \right)^{2g-2}.$$

Bryan and Pandharipande's Deformation of Dijkgraaf's TQFT

In [BP2], Jim Bryan and Rahul Pandharipande have been able to give a TQFT structure to the local Gromov-Witten theory of curves embedded in a Calabi-Yau threefold. This TQFT (that we will call \mathcal{BP}) is a one parameter deformation of \mathcal{D} : the ground field \mathbb{C} is replaced by the formal power series ring $\mathbb{C}[[t]]$; the coefficients of \mathcal{BP} are generating functions for local Gromov-Witten invariants, that evaluated at $t = 0$ reduce to the corresponding Hurwitz numbers in \mathcal{D} . This brilliant insight reduces the complete study of local GW theory to the computations of the local invariants for the one-pointed and the three-pointed spheres (disc and pair of pants in the topological setting).

In [BP2] the one pointed invariants are computed by using virtual localization to express the Gromov-Witten invariant as a Hodge integral that was previously computed by Faber and Pandharipande in [FP].

The three-pointed invariants however are much harder to compute, since localization techniques cannot be used (we can't put a torus action on a three punctured sphere!).

The problem is finally solved, again brilliantly, in [BP3]: the whole theory is further generalized to the study of the equivariant Gromov-Witten theory of curves X embedded in an open threefold, not necessarily Calabi-Yau. The open threefolds considered are rank 2 vector bundles $N = L_1 \oplus L_2$, with L_1 and L_2 line bundles of degree k_1, k_2 on X .

The theory so obtained is a "weighted" TQFT: cobordisms come equipped with a pair of integers, (k_1, k_2) , that are called levels. When concatenating two cobordisms, levels are added componentwise.

In this setting the three-pointed invariants can be computed in level $(0, 0)$. One special invariant is reduced to a Hodge integral on the hyperelliptic locus $\overline{Hyp} \in \overline{M}_h$. All other coefficients are obtained recursively from this one and lower degree coefficients.

My Work

I started being interested in the work of Bryan and Pandharipande three years ago. After about one year Jim Bryan suggested that I try to parallel their theory in a different setting, i.e., replacing Gromov-Witten invariants, that are intersection numbers on moduli spaces of Stable Maps, with analogous intersection numbers on moduli spaces of Admissible Covers. This has become my thesis project. The theory has revealed to be closely connected with \mathcal{BP} and yet different enough to be very interesting. Also, a wealth of questions and projects have opened up that I would love to pursue in the future.

Admissible Covers

Moduli Spaces of Admissible covers are a "natural" compactification of the Hurwitz scheme. The fundamental idea is that, in order to understand limit covers, we allow the base curve to degenerate together with the cover. Branch points are not allowed to "come together"; as two or more branch points tend to collide, a new component of the base curve sprouts from the point of collision, and the points transfer onto it. Similarly, upstairs the cover splits into a nodal cover.

Moduli spaces of Admissible Covers were introduced originally by Harris and Mumford in [HM]. They failed to become popular for quite a while, mostly because they are in general not normal, even if the normalization is always smooth. Only recently in [ACV], Abramovich,

Corti and Vistoli exhibit this normalization as the stack of balanced stable maps of degree 0 from twisted curves to the classifying stack \mathcal{BS}_d . This way they attain both the smoothness of the stack and a nice moduli-theoretic interpretation of it.

We abuse notation and refer to the Abramovich-Corti-Vistoli spaces as Admissible Covers.

We denote by

$$\overline{Adm}_{h \xrightarrow{d} X, (\underline{\eta})}$$

the moduli space of Admissible covers of degree d of a pointed curve (X, p_1, \dots, p_n) by curves of genus h , subject to ramification conditions specified by the vector of partitions $\underline{\eta}$

Moduli spaces of Admissible Covers seem to have the following advantages over Stable maps:

1. Admissible Covers are smooth stacks.
2. All boundary components are “of the right dimension”.
3. All irreducible components of a nodal admissible cover come equipped with a degree d map. In some sense, Admissible Covers keep track of “ d -gonality” also over the boundary.

The (weighted) TQFT of Admissible Covers

The setup

Like in \mathcal{BP} , we start from a smooth marked curve (X, p_1, \dots, p_n) , and a rank two vector bundle N on it that splits as the direct sum of two line bundles L_1 and L_2 of degree k_1 and k_2 .

Let $\underline{\eta} = (\eta_1, \dots, \eta_n)$ be a vector of partitions of the integer d , that we interpret as ramification data over the marked points.

A two dimensional torus T acts on N by scaling the fibers of the line bundles. Denote $\mathbb{C}[s, t]$ the T equivariant cohomology ring of a point.

Now, we define

$$AdmGW_{d[X], \underline{\eta}}^h(k_1, k_2) = \int_{[\overline{Adm}_{h \xrightarrow{d} X, (\underline{\eta})}]} e(-R^\bullet \pi_* f^*(L_1 \oplus L_2)),$$

where:

- integration means equivariant push-forward to the class of a point.
- e is the equivariant euler class of the virtual bundle in question.
- π is the universal family over the space of Admissible Covers.
- f is the universal cover map from the universal family all the way down to X .¹

Finally, we organize these invariants in generating function form:

$$AdmGW_{d[X], \underline{\eta}}(k_1, k_2) := \sum u^{2h+\star} AdmGW_{d[X], \underline{\eta}}^h(k_1, k_2),$$

where \star is a natural shift depending on $d, g(X), k_1, k_2$ and n .

¹a priori the universal map is defined into an artin stack, but then a contraction map can get us all the way down to X .

Project 2. *Prove the conjecture for general degree d . The localization argument I have used unfortunately quickly gains combinatorial complexity.*

It is remarkable that these coefficients for the Calabi-Yau cap are strictly related to the coefficients in the \mathcal{BP} theory. In fact the Admissible Cover theory is a normalization of \mathcal{BP} by the (non trivial) coefficient of the identity.

Project 3. *Understand this relationship. Can the ratio be interpreted as excess intersection?*

Using localization it's possible to obtain relations among our one pointed coefficients and various types of Hurwitz numbers on \mathbb{P}^1 ; even though this doesn't fit precisely into this theory I would be interested to explore further these relations.

Project 4. *Use these relations "backwards", to try and obtain generating functions for various (not only simple) Hurwitz numbers of \mathbb{P}^1 .*

Specialization to the anti-diagonal action of \mathbb{C}^*

Looking at the anti-diagonal embedding of \mathbb{C}^* inside T amounts to imposing the condition $t = -s$ on the equivariant parameters. In this case the level $(0,0)$ multiplication simplifies dramatically, reducing essentially to the multiplication in the class algebra of the symmetric group S_d , where the semisimple basis is classically known. It's then possible to completely diagonalize the theory (i.e. to express it as a direct sum of one dimensional TQFT's), and to give closed formulas for the invariants of un-marked curves.

Theorem: for X a smooth curve of genus g , in the anti-diagonal action specialization:

$$AdmGW_{d[X]}(k_1, k_2) = (-)^{d(g-1-k_2)} s^{d(2g-2-k_1-k_2)} \sum_{\rho} \left(\frac{d!}{dim \rho} \right)^{2g-2} \left(\frac{dim \rho}{dim_Q \rho} \right)^{k_1+k_2} Q^{n(\rho)k_1+n(\rho')k_2},$$

where

- ρ are irreducible representations of S_d ;
- ρ' is the conjugate representation to ρ ;
- $n(\rho) = \sum_1^{\ell(\rho)} (i-1)\rho_i$, where ρ is identified with its associated partition;
- $Q = e^{iu}$;
- $dim_Q \rho = d! \prod \frac{1-Q}{1-Q^{hooklength(\rho)}}$.

If we set $Q = 1$, then $dim_Q \rho = dim \rho$ by the hooklength formula. Thus our TQFT is a deformation of Dijkgraaf's.

Project 5. *Understand the relationship with the representation theory of the symmetric group S_d . Does the Q -dimension of an irreducible representation ρ have a representation theoretic meaning?*

Specialization to the diagonal action of \mathbb{C}^*

Looking at the diagonal embedding of \mathbb{C}^* inside T amounts to imposing the condition $t = s$ on the equivariant parameters. This is the scenario originally studied by Bryan and Pandharipande in [BP2]. However, the theory seems in this case to be more complicated, and so far, has been completely solved, both in the \mathcal{BP} and in the Admissible cover settings, only for degrees 1 and 2. An arbitrary number of coefficients for un-marked curves can be computed in degree 3 as well, but already in degree 3 the task of diagonalizing the theory seems not to be underestimated. I am conjecturing that a new change of coordinates may be necessary in order to do so.

Project 6. *Try to diagonalize the degree 3 theory. At least, understand if the problem is approachable, or if there are obstructions to obtaining a closed formula for the theory like in the case of degrees 1 and 2.*

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