University of Utah
Department of Mathematics

TOPOLOGICAL QUANTUM FIELD THEORIES AND THE GOPAKUMAR-VAFA CONJECTURE

Research Proposal

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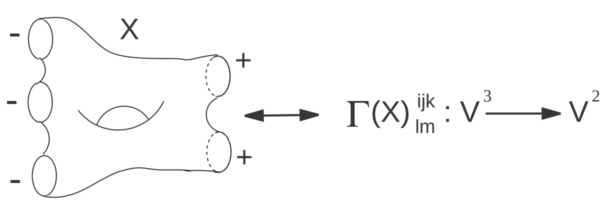
My main area of research is algebraic geometry. In particular I am interested in Topological Quantum Field Theories (TQFT's). These are constructions that have their origin in physics, but are described by an elegant mathematical theory. To a physical system is associated, at any instant of time, the vector space of the states of the system. Time then allows the system to evolve, and such evolution is described by the Feynman operators. Mathematically, these are maps between the various state spaces. The main problem of physics is, in some very broad and oversimplified sense, to understand these evolution operators so as to explain physical phenomena and make predictions about them. TQFT's offer a very simpified model for "rehearsing" computations that we wish to reproduce in much broader generality. Besides being such a handy tool for physicists, TQFT's distill to beautiful theories that branch out to several different areas in mathematics, such as algebraic geometry, topology, group theory and representation theory, and offer a convenient and insightful way to give structure to interesting geometric invariants. This interconnection of many disciplines is not only an extremely fascinating philosophical phenomenon; it is in general a most productive and useful tool, as it allows us to "translate" physical problems into geometric and algebraic ones, or vice versa. And in science it is well known that often seemingly impossible problems become solvable if viewed in the right way!

Two dimensional Topological Quantum Field Theories.

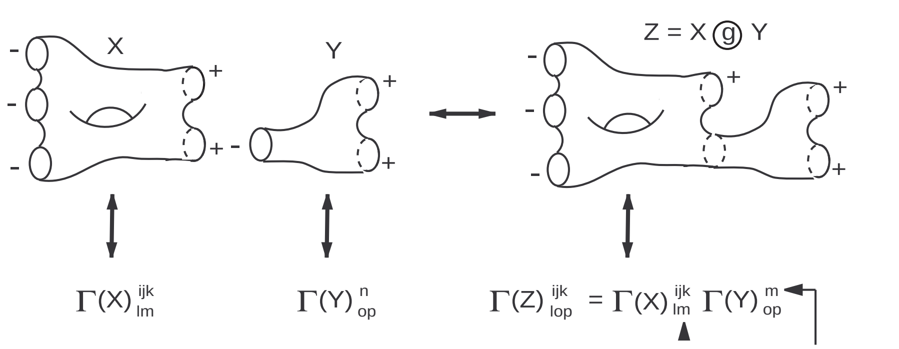
A two dimensional topological quantum field theory is a function:

$$\left\{ \begin{array}{c} \textit{Topological surfaces } (= \textit{ dimension } 2) \\ \textit{with oriented boundary components} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{tensors associated to a} \\ \textit{fixed vector space } V \end{array} \right\}$$

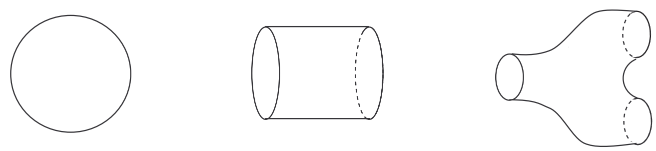
This means that to a topological surface X with p positively oriented and n negatively oriented boundary circles we assign a tensor of type (p, n), that we can think of as a multilinear map from V^n to V^p. For example



Of course this assignment can't be just random. We have an operation on surfaces that consists of gluing a positive boundary circle with a negative one to obtain a new surface. We want this operation to correspond, on the other side, to the natural contraction of tensors (using Einstein's notation, this just means summing over the repeated indices).



Since every surface with boundary, however complex, can be decomposed into discs, annuli and pairs of pants, it follows that once we know the tensors associated to these basic "building blocks", we are able to determine (uniquely!) the tensor associated to any surface, by following the above prescription.



disc annulus pair of pants

The physical interpretation of this construction can be roughly sketched as follows: imagine 3D space sliced in a sequence of parallel planes, whose normal direction represents the time axis. At any instant (i.e., at any plane) the physical state of the system is represented by the configuration of curves obtained by slicing the surface with the plane. Hence the negative boundary components represent space at the initial instant of observation, while the positive boundary components represent space at the terminal instant; the surfaces represent space-time. The associated vector spaces are the state spaces and the tensors between them are the time-evolution operators (Feynman path integrals). Then the gluing conditions just correspond to the elementary fact that if we observe a system from "time A" to "time B", and then from "time B" to "time C", we should get the same evolution from time "A" to time "C" that we would observe forgetting about the intermediate "time B".

TQFT's are interesting to topologists because they provide invariants of manifolds, i.e. ways to tell two different manifolds apart. This is not of much interest in dimension 2, as we already have a complete and satisfactory classification of closed topological surfaces; but already for dimension 3 it is a completely different story, and it is recent news that a proposed solution to the problem by Perelman is currently being evaluated, with a million dollars from the Clay Institute still pending.

But the realm of 2 dimensions hides a very pleasant surprise for algebraic geometers. A two dimensional TQFT endows the vector space V with the structure of a commutative Frobenius Algebra. This means we have a way of multiplying two vectors, and such an operation is commutative, associative and has an identity. Furthermore, we have a scalar product that is compatible with multiplication. It is fairly easy to visualize this; for example, multiplication is nothing but a bilinear map:

$$\begin{array}{rccc} m: & V \times V & \longrightarrow & V \\ & v_1, v_2 & \mapsto & v_1 \cdot v_2 \end{array}$$

The TQFT structure provides a unique candidate for such a map: the tensor associated to the pair of pants.

Feynmann graph: two particles merging

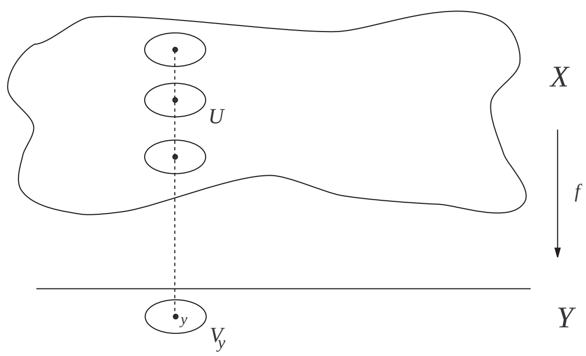
Pair of pants: multiplication

What is even more remarkable is that this correspondence between TQFTs and Frobenius Algebras is actually a bijection: given a TQFT, I can associate to it a unique Frobenius Algebra and vice versa. And if I do the two processes one after the other I just end up where I started.

Dijkgraaf's TQFT of Covers.

In the early 1990's, the mathematical physicist Robbert Dijkgraaf (University of Amsterdam) noticed that a TQFT approach yielded a beautiful and elegant solution to a classical mathematical problem: counting ramified and unramified covers of a surface.

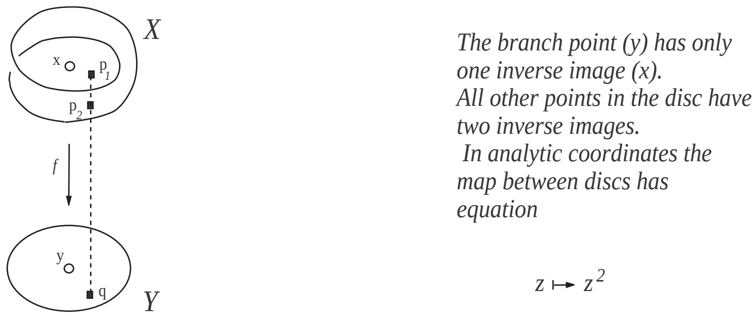
A map of surfaces $f : X \to Y$ is a *cover of degree* d if around each point $y \in Y$ we can find a disc whose inverse image is a disjoint union of d discs mapping homeomorphically to it. In a picture:



A cover is ramified if over a finite number of points of Y we allow the map between the discs to be represented, in local analytic coordinates, by the equation

$$w = z^k.$$

EXAMPLE: A RAMIFICATION POINT OF ORDER 2.



Dijkgraaf thinks of a surface X with boundaries as of a punctured surface. You can visualize this by shrinking the boundary circles untill they "look like" points.

Then he assigns to X a tensor whose coefficients are the number of degree d covers of X, allowed to ramify only over the points P_i.

It is quite amazing that these coefficients arrange themselves to produce a TQFT structure. What is even more amazing is that the corresponiding Frobenius Algebra is $\mathcal{Z}(\mathbb{Q}[S_d])$, the center of the group ring of the permutation group in d elements. This is an extremely familiar object to algebraists and has many good properties. In particular, $\mathcal{Z}(\mathbb{Q}[S_d])$ is a *semisimple* algebra, i.e., it has a basis $\{e_1, \ldots, e_n\}$ such that the multiplication reduces to:

$$e_i \cdot e_j = 0 \text{ if } i \neq j$$

$$e_i \cdot e_i = e_i.$$

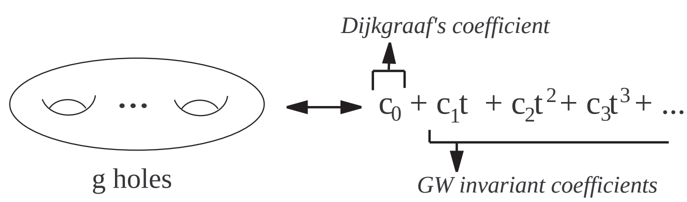
This observation allowed Dijkgraaf to write an elegant formula counting unramified degree d covers of closed surfaces of arbitrary genus g:

$$\sum_R \left(\frac{dim(R)}{d!} \right)^{2g-2}.$$

The sum is over all irreducible representations of the finite group S_d, and here is yet another branch of mathematics, representation theory, that comes to be an important player of this game.

Bryan and Pandharipande' s Deformation of Dijkgraaf's TQFT.

In the last five years Jim Bryan (UBC) and Rahul Pandharipande (Princeton) have picked up these ideas and taken them to even further depth. They were able to view Dijkgraaf's coefficients as just the initial terms of infinite power series, whose coefficients of higher order terms are *Gromov-Witten invariants* of complex curves embedded in a Calabi Yau threefold. These are subtle and sophisticated invariants of curves that have arisen from the recent intertwining of string theory with algebraic geometry, and contain significant enumerative information about the curve. Edward Witten, a physicist considered by many the "Pope" of modern string theory, and recipient of the Fields Medal in 1990 (the highest honor in mathematics) is one of the founding fathers of this theory. In 1998 Maxime Kontsevich was also awarded a Fields Medal for further developing Gromov-Witten theory. Unfortunately, these invariants are, in general, extremely difficult to compute. However Bryan and Pandharipande are telling us that they are organized in a TQFT structure, at the price of substituting the initial vector space V with a more complicated object, namely a free module over the power series ring $\mathbb{Q}[[t]]$. Hence if we are able to compute the invariants relative to the three building blocks (disc, annulus, pair of pants), we can produce all others with minimal effort.



Furthermore, since by setting $t = 0$ we recover Dijkgraaf's TQFT, this new TQFT automatically inherits semisimplicity. This is very promising, since calculations in semisimple algebras simplify dramatically. The problem is, as of now, there is not a way to concretely find out the semisimple basis, except with ad hoc techniques in very low degrees.

The Gopakumar Vafa Conjecture.

In 1998 two theoretical physicists, Rajesh Gopakumar and Cumrun Vafa (Harvard), conjectured a relationship between Gromov-Witten invariants and counts of certain BPS states in M theory (one of the most recent and sophisticated incarnations of string theory).

The formula that they produce is terrifyingly long and complicated. However, the conjecture turns out to be equivalent to a statement that is equally terrifying, but because of its beauty and simplicity:

after the change of variable

$$w = 2\sin\left(\frac{t}{2}\right),$$

the power series associated by the Bryan-Pandharipande TQFT to closed surfaces reduce to polynomials in w with integer coefficients.

I will never be able to stress enough how mind boggling this is: Gromov Witten invariants are usually fairly ugly looking rational numbers; we are adopting a transcendental change of variable; there is no mathematical reason that would suggest such a dramatic simplification to be plausible. We are yet again confronted with a phenomenon that seems to be happening more and more often: it is the physics, i.e., in some sense, our universe, that is giving us insights on amazingly sophisticated mathematical truths.

My project.

The obvious ultimate goal would be to prove the Gopakumar Vafa conjecture, or at least some positive result in that direction.

Currently, with my advisor, Professor Aaron Bertram, I am involved in two projects that we are hoping will lead us down the right path:

- I am trying to exploit the TQFT structure, in particular the constraints given by the associativity of multiplication, to be able to determine all the coefficients of the Bryan-Pandharipande TQFT, given a fairly limited set of initial conditions. I am investigating a simple algorithm that in degree 3 should produce the multiplication (or equivalently, the pair of pants) coefficients of the TQFT up to a finite choice. It seems that even the "wrong" excess coefficients that this method may produce could be interesting to observe, as they may give us insights into symmetries or geometrical aspects of the TQFT structure.

- I am planning to construct another deformation of Dijkgraaf's TQFT, that encodes Gromov-Witten invariants of admissible covers of curves. According to the intuition of both my advisor and Jim Bryan, with whom I am in contact, such invariants should be more naturally related to the Gopakumar Vafa BPS counts. Furthermore, we know a priori that admissible cover invariants will be integral; a reasonable hope is to unveil the mysterious change of coordinates $w = 2\sin(t/2)$, and understand its geometrical meaning.

Conclusions.

I am standing on the edge of a mysterious and fascinating territory. It is quite extraordinary to see so many different branches of mathematics interact together, all apparently orchestrated by the world of physics, which is, after all, our real world! Many open questions and serious technical difficulties lie in front of me; I am looking forward to being able to make my own small contribution to the understanding of these topics.

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