Yet More Power Series

Math 1220 (Spring 2003)

Not only can we "do calculus" on power series. We can also "do algebra."

Algebraic Properties of the Power Series Functions: Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

then:

(i)

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

as we can see by adding them as you would add polynomials:

$$\begin{array}{ccccccccc} & a_0 & + & a_1 x & + & a_2 x^2 & + & \cdots \\ + & b_0 & + & b_1 x & + & b_2 x^2 & + & \cdots \\ \hline & (a_0 + b_0) & + & (a_1 + b_1)x & + & (a_2 + b_2)x^2 & + & \cdots \end{array}$$

(ii)

$$f(x)g(x)$$

is seen by multiplying the power series as you would multiply polynomials.

$$\begin{array}{cccccccccc} & a_0 & + & a_1 x & + & a_2 x^2 & + & \cdots & & \\ \times & b_0 & + & b_1 x & + & b_2 x^2 & + & \cdots & & \\ \hline & a_0 b_0 & + & a_1 b_0 x & + & a_2 b_0 x^2 & + & \cdots & & \\ & & & a_0 b_1 x & + & a_1 b_1 x^2 & + & a_2 b_1 x^3 & + & \cdots \\ & & & & & a_0 b_2 x^2 & + & a_1 b_2 x^3 & + & \cdots \\ & & & & & & & \vdots & & \\ \hline & (a_0 b_0) & + & (a_1 b_0 + a_0 b_1)x & + & (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 & + & \cdots & & \end{array}$$

(iii) The reciprocal of $\sum_{k=0}^{\infty} b_k x^k$ is another power series whenever $b_0 \neq 0$. You get the reciprocal:

$$\frac{1}{\sum_{k=0}^{\infty} b_k x^k} \quad \text{by solving} \quad \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = 1$$

Example: To find the power series for

$$\frac{1}{1-x+x^2}$$

we solve:

$$\begin{array}{cccccccccc}
 & a_0 & + & a_1x & + & a_2x^2 & + & a_3x^3 & + & \cdots \\
\times & 1 & - & x & + & x^2 & & & & \\
\hline
 & a_0 & + & a_1x & + & a_2x^2 & + & a_3x^3 & + & \cdots \\
 & & & -a_0x & - & a_1x^2 & - & a_2x^3 & - & \cdots \\
 & & & & & a_0x^2 & + & a_1x^3 & + & \cdots \\
\hline
 & 1 & & & & & & & &
\end{array}$$

Now we solve for all the a_k's (or rather as many as we feel like).

$a_0 = 1$

$a_1 - a_0 = 0$ gives $a_1 = 1$

$a_2 - a_1 + a_0 = 0$ gives $a_2 = 0$

$a_3 - a_2 + a_1 = 0$ gives $a_3 = -1$

$a_4 - a_3 + a_2 = 0$ gives $a_4 = -1$

$a_5 - a_4 + a_3 = 0$ gives $a_5 = 0$

and if you care to go further, you will eventually find a pattern developing. But we will be happy with writing:

$$\frac{1}{1-x+x^2} = 1 + x - x^3 - x^4 + ...$$

and knowing that we could furnish more terms if they were needed.