Taylor Series

Math 1220 (Spring 2003)

Taylor series are generalized power series that look like:

$$\sum_{k=0}^{\infty} a_k (x-a)^k$$

where a is a fixed real number. If $a = 0$, this is an ordinary power series. But even if $a \neq 0$, this isn't any harder to deal with. Here's what happens:

The Convergence Set: In this case the convergence set is one of:

$$(-R+a, R+a), [-R+a, R+a), (-R+a, R+a], [-R+a, R+a]$$

$$\text{or } (-\infty, \infty) \text{ or } \{a\}$$

and we find the convergence set in the usual way. Namely, the ratio test tells us that:

$$\frac{1}{R} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

and then we plug in the endpoints to see if they converge.

Taylor Series: If we have a function $f(x)$ and a is in the domain of $f(x)$, then the **Taylor series** for $f(x)$ around a is the series:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

and the coefficients a_k are determined just like the Maclaurin coefficients:

$$a_0 = f(a),\ a_1 = \frac{f'(a)}{1!},\ a_2 = \frac{f''(a)}{2!},\ \ldots$$

or, in other words, if there is a Taylor series around a, then it is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

and we can estimate how good these Taylor series approximate $f(x)$ with:

Taylor's Theorem with Remainder:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(a)(x-a)^n + R_n(x)$$

where the error term is:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between $a - x$ and $a + x$.

Example: The Taylor series for

$$f(x) = \frac{1}{1+x}$$

around $x = 1$ is gotten by taking derivatives:

$$f(1) = \frac{1}{2},\ f'(x) = -\frac{1}{(1+x)^2}$$

$$f'(1) = -\frac{1}{4},\ f''(x) = \frac{2}{(1+x)^3}$$

$$f''(1) = -\frac{2}{8},\ f'''(x) = -\frac{3!}{(1+x)^4}$$

$$\vdots$$

$$f^{(n)}(1) = (-1)^n \frac{n!}{2^{n+1}}$$

so the coefficients of the Taylor series are:

$$a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n (\frac{1}{2})^{n+1}$$

and putting this all together we get:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k (\frac{1}{2})^{k+1} (x-1)^k$$

which has $R = 2$ and convergence set $(1-2, 1+2) = (-1, 3)$.