

Infinite Series

Math 1220 (Spring 2003)

An infinite **series** is the sum of the terms of an infinite sequence:

$$a_1 + a_2 + a_3 + a_4 + \dots$$

The right way to think about this is as the sequence of the **partial sums**:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \dots$$

and people like to give these partial sums their own names:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

etc

And there is some fancy notation so we don't have to write lots of symbols:

$\{a_k\}$ stands for the **sequence** $a_1, a_2, a_3, a_4, \dots$

$\sum_{k=1}^{\infty} a_k$ stands for the **series** $a_1 + a_2 + a_3 + a_4 + \dots$

and

$S_n = \sum_{k=1}^n a_k$ stands for the **partial sum** $S_n = a_1 + a_2 + \dots + a_n$

Definition: When the sequence of partial sums of a series converges to L . That is, when:

$$\lim_{n \rightarrow \infty} S_n = L$$

then say that the series *converges to* L and we write:

$$\sum_{k=1}^{\infty} a_k = L$$

If the sequence of partial sums diverges, we say that the series diverges.

Example 1. Geometric Series: These are series of the form:

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \dots$$

The partial sums of this series are easy to calculate using a trick:

$$S_n - rS_n = (a + ar + \dots + ar^{n-1}) - (ar + ar^2 + \dots + ar^n) = a - ar^n$$

(assuming that $r \neq 1$) so that:

$$S_n = \frac{a - ar^n}{1 - r}$$

So:

If $|r| < 1$, then the series converges to:

$$\lim_{n \rightarrow \infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r}$$

But if $|r| \geq 1$, then the series diverges.

A No-Brainer Test for Divergence: If the **sequence** $\{a_k\}$ diverges, or

$$\lim_{k \rightarrow \infty} a_k \neq 0$$

then the **series** $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: Consider:

$$S_n - S_{n-1} = a_n$$

so in order for the sequence of S_n 's to converge, it must be the case that the sequence of the a_n 's converges to zero.

Warning! You can not turn this no-brainer into a test for **convergence!** There are plenty of sequences $\{a_k\}$ that converge to zero, but for which the series $\sum_{k=1}^{\infty} a_k$ do not converge!

Example 2. The Harmonic Series. The series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, even though obviously:

$$\lim_{k \rightarrow \infty} \frac{1}{k} = 0$$

Here's why. We can group the terms of the harmonic series:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

in groupings of 1, 1, 2, 4, 8, 16, 32, Then each of the sums inside the parenthesis is $\geq \frac{1}{2}$, so the overall sum is greater than $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$ so it cannot have a finite limit!

Example 3. Collapsing Series. Here is one more series that does converge, and for which we can explicitly give the limit.

Suppose $\{a_k\}$ is a sequence with a formula that can be written:

$$a_k = f(k) - f(k - 1)$$

Then the partial sums collapse upon each other:

$$S_1 = f(1) - f(0), S_2 = (f(1) - f(0)) + (f(2) - f(1)) = f(2) - f(0)$$

$$S_3 = (f(3) - f(2)) + (f(2) - f(1)) + (f(1) - f(0)) = f(3) - f(0)$$

and following the same pattern:

$$S_n = f(n) - f(0)$$

so that in this special example, if:

$$\lim_{k \rightarrow \infty} f(n) = 0$$

then the series $\sum_{k=1}^{\infty} a_k$ converges. In fact, it converges to:

$$\sum_{k=1}^{\infty} a_k = -f(0)$$

SubExample: Let:

$$a_k = \frac{1}{(k+1)(k+2)} = -\frac{1}{k+2} - \left(-\frac{1}{k+1}\right)$$

so in this subexample, we should let:

$$f(k) = -\frac{1}{k+2}$$

and then $a_k = f(k) - f(k-1)$, so we know immediately that:

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = -\left(-\frac{1}{0+2}\right) = \frac{1}{2}$$

which isn't immediately obvious!