

## Infinite Sequences

Math 1220 (Spring 2003)

A sequence is one real number followed by another one forever:

$$a_1, a_2, a_3, \dots$$

Many sequences have a pattern, which allows us to infer the rest:

$$1, 3, 5, 7, \dots$$

is clearly the sequence of odd natural numbers, which has a formula:

$$a_n = 2n - 1$$

Here are some more:

- $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$  has the formula  $a_n = 1 - \frac{1}{n}$
- $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$  has the formula  $a_n = 1 - \frac{1}{2^{n-1}}$
- $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$  has the formula  $a_n = (-1)^{n-1} \frac{1}{n}$

**Definition:** The sequence  $a_1, a_2, a_3, \dots$  **converges** to  $L$  if for every  $\epsilon > 0$ , we can find an integer  $N$  such that:

$$|a_n - L| < \epsilon \text{ whenever } n \geq N$$

and in that case we write:

$$\lim_{n \rightarrow \infty} a_n = L$$

The sequence **diverges** if it doesn't converge to any limit  $L$ .

**A useful fact:** If  $a_n = f(n)$  is a formula for the sequence  $a_1, a_2, a_3, \dots$  and:

$$\text{if } \lim_{x \rightarrow \infty} f(x) = L \text{ then } \lim_{n \rightarrow \infty} a_n = L$$

so we can use limits of functions to deduce limits of sequences. For example, we can use l'Hôpital's rule when it is useful!

**Another way to write a sequence:** It is useful shorthand to write:

$$\{a_n\} \text{ for the sequence } a_1, a_2, a_3, \dots$$

**Some more useful facts:**

**1. The Squeeze.** If  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are sequences with:

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N \text{ and } \lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$$

then

$$\lim_{n \rightarrow \infty} b_n = L$$

Literally, the sequence  $\{b_n\}$  is eventually squeezed between  $\{a_n\}$  and  $\{c_n\}$ .

**2. Hitting the Ceiling.** If  $\{a_n\}$  is a **nondecreasing sequence**, which means:

$$a_1 \leq a_2 \leq a_3 \leq \dots$$

and if  $\{a_n\}$  has an **upper bound**, that is:

$$a_n \leq U \text{ for all } n$$

then  $\{a_n\}$  has a limit which is (obviously) less than or equal to  $U$ .

Similarly, if  $\{a_n\}$  is a **nonincreasing sequence** with a lower bound, then it has a limit, which is greater than (or equal to) the lower bound. This would be hitting the floor(!)