Gran Finale

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CALCULUS II COURSE, FALL 2003

Plan:

1. A tiny little bite of multivariable calculus.
2. An alternative way to coordinatize the plane: POLAR.
3. A fancy way to do calculus: DIFFERENTIAL FORMS.

INTRO:

Throughout this semester we have seen a lot of new mathematical concepts; a lot of them, alas, quite technical and not necessarily utterly pleasant or thrilling. Yet I have asked you to believe that we have been building the foundations for developing math to the point of being interesting, and APPLICABLE to many different problems. In these last few lectures I want to make a little and "informal" incursion into more advanced material, with the hope of giving you a flavor (and hopefully a pleasant one) of what lies ahead.

At this point, with Taylor solidly in your hands, we could be heading in several directions, many of which already full of "real life" applications. However, trusting that you'll have many chances for applications in the prosecution of your studies, I chose to remain in the realm of mathematics, and present a few things that I know and like.

After spending so much time worrying about functions of one variable, learning how to take derivatives and integrate, it would be unfulfilling if we were to limit ourselves to one dimension. After all, most experiments, business situations, etc. etc. naturally depend on several parameters. Luckily, calculus extends in some sort of a "<u>natural</u>" way to this setting. We will see how to think of functions of many variables, and how to deal with integration & differentiation.

In doing so it will turn out to be useful to be able to "view" the plane or 3-dimensional space in a slightly different way, and that's where "<u>polar coordinates</u>" come into play.

Finally, I want to show you, in a forcefully imprecise and "handwavy" manner, (but hopefully "operative" enough to allow you to work with it), a nice and "high" tech way of dealing with multivariable differentiation and integration. We'll have to learn just a few bizarre-looking axioms, but then we'll get for free all the differential identities so painfully learned in physics; moreover, we'll be able to integrate in whatever coordinate system, with no mental effort but some bookkeeping.

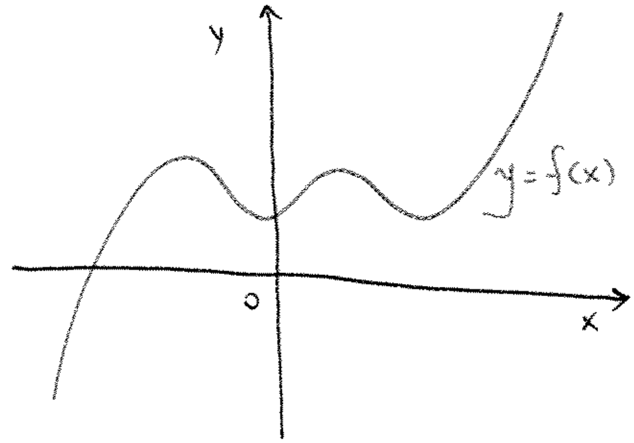
So without further, adó, let's get started!

§ 1. Multivariable Calculus

We have so far been dealing with functions: $\boxed{y = f(x)}$

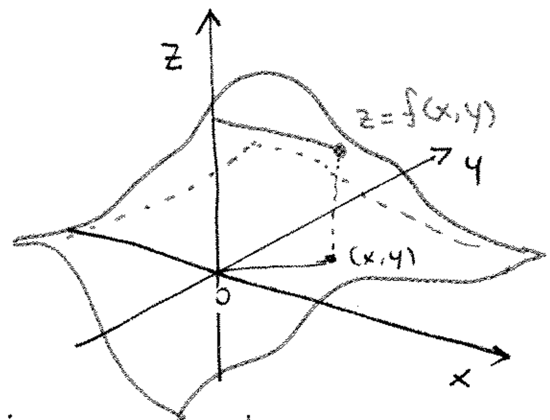
$$f : \underset{\text{"}x\text{"}}{\underset{\uparrow}{\mathbb{R}}} \longrightarrow \underset{\text{"}y\text{"}}{\underset{\uparrow}{\mathbb{R}}}$$

And represented them by looking at their graph in the xy plane:

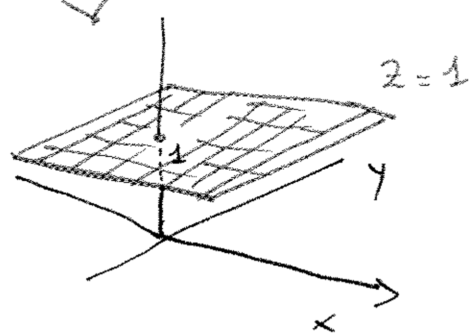


Now we will be dealing with functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, where n can be any number. But let's start slow:

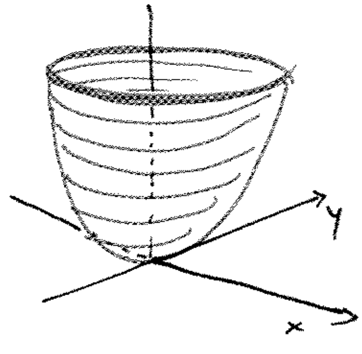
A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function whose inputs are points in the plane (or, alternatively, pairs of numbers (x, y)) and whose outputs are real numbers. Its graph will now be a surface in 3-dim'l space:



Examples: • $z = 1$ – a horizontal plane:



• $z = x^2 + y^2$ – a paraboloid



But also much nastier things:

- $z = \sin x + \cos y + 3xy$
- $z = \dfrac{e^{xy}}{[illegible]}$

Exercises: 1) sketch the graph of the following functions:

- $z = x$
- $z = \sqrt{1 - x^2 - y^2}$

2) find the domain and the range of the following functions:

- $z = \sqrt{1 - x^2 - y^2}$
- $z = \dfrac{4x + 3y}{2x + y + 1}$
- $z = \ln(x^2 + y^2 - 1)$

Now, there is nothing special about dimension 2; we can now define a function from $\mathbb{R}^n \to \mathbb{R}$ just by increasing the number of input variables. Of course the problem is, it's impossible to visualize such graphs, as we can't draw in 4 dimensions or higher. That's why we'll stick to dimension two for this little presentation. But everything carries on exactly the same.

E.g.: $f: \mathbb{R}^3 \to \mathbb{R}$ $\qquad w = 3x^2 + 4xy + z^3 + 2$

Q: what is $f(0)$?

$g: \mathbb{R}^3 \to \mathbb{R}$ "temperature in this room"

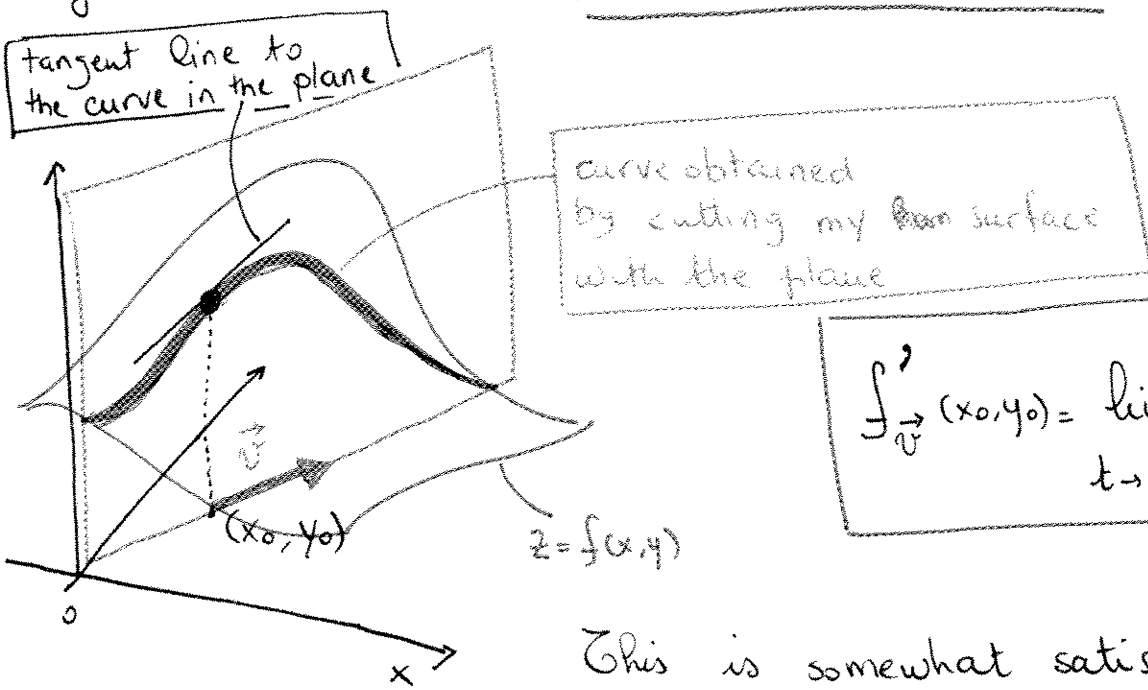
$h: \mathbb{R}^4 \to \mathbb{R}$ $\qquad y = x_1x_2 - x_1x_3 + x_2x_4 - x_4^2$

Q2: what is $h(0)$?

$h(1)$?

Now we want to learn how to take $\boxed{\text{derivatives}}$

So let us place ourselves at the point (x_0, y_0) in the plane. If we "slice" our surface with a vertical plane in a given direction $\vec{v}$ we obtain a curve in that plane, that we can think as of the graph of a 1-dim'l function. The derivative of such function at the point (x_0, y_0) is called the directional derivative.



Suppose $\|\vec{v}\| = 1$

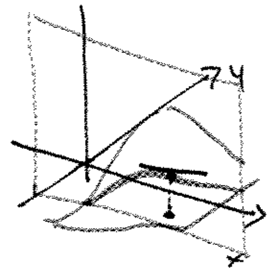
$$\boxed{f'_{\vec{v}}(x_0,y_0) = \lim_{t\to 0} \frac{f((x_0,y_0)+t\vec{v}) - f(x_0,y_0)}{t}}$$

This is somewhat satisfactory. Given any direction in the plane, we are able to determine the rate of growth of my function in that particular direction. But we can do even better!

There are two "special" directions in the plane, i.e. the directions of the x and the y axes. They are so special that we give to the directional derivatives in these directions special names:

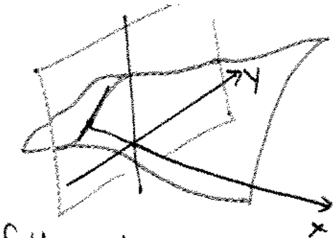
PARTIAL DERIVATIVE with respect to x

$\frac{\partial f}{\partial x}$: the directional derivative in the x-direction



PARTIAL DERIVATIVE with respect to y

$\frac{\partial f}{\partial y}$: the directional derivative in the y direction



How to compute them: just use the following trick: to do $\frac{\partial f}{\partial x}$, think of f as a function of only x by thinking y to be a constant, and take the "regular" derivative of f. Similarly for $\frac{\partial f}{\partial y}$, think of x as a constant.

Example: $f(x,y) = x^2 + y^3 + 2x^2y$

$$\frac{\partial f}{\partial x} = 2x + 4xy$$

$$\frac{\partial f}{\partial y} = 3y^2 + 2x^2$$

Exercise: compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions:

1) $f(x,y) = x^2 + y^2 - 1$

2) $f(x,y) = e^{x^2 y} + \ln(x^2+y)$

3) $f(x,y) = \frac{x^2}{y} + \sin(xy)$

Cool fact: if I know the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ I can easily compute any directional derivative.

Say $\vec{v} = (v_1, v_2)$ is a vector of length one, then

$$f'_{\vec{v}}(x_0, y_0) = \vec{\nabla} f \cdot \vec{v} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\partial f}{\partial x}(x_0,y_0) \cdot v_1 + \frac{\partial f}{\partial y}(x_0,y_0) \cdot v_2$$

The vector $\begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$ is also important enough to deserve

a name and a symbol of its own.

$$\boxed{\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \partial f/\partial y \end{bmatrix} = \underline{\text{GRADIENT of } f}}$$

Example: Find the directional derivative of $z = x^2 + y + 1$

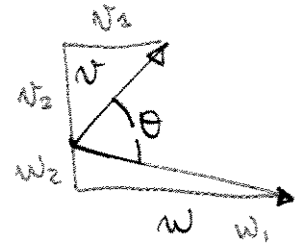
at the point $(0,1)$ in the direction 

SOLUTION: The unit vector in that direction is $\underline{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\nabla_{(0,0)} f = \begin{pmatrix} 2x \\ 1 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f'_v(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = 0 \cdot \frac{\sqrt{2}}{2} + 1 \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

Supercool fact: just like the derivative is related to the tangent line to the graph of the function, the gradient is related to the tangent plane to the graph of the function: to see this let us first recall a few things about the $\bullet$ product of vectors:



$$\underline{v} \bullet \underline{w} = v_1 w_1 + v_2 w_2 = |v||w| \cos\theta$$

Hence $\vec{\underline{v}} \bullet \vec{\underline{w}} = 0$ if and only if $\underline{v}$ and $\underline{w}$ are perpendicular.

It turns out (excellent exercise: try and figure out why!!) that $\begin{bmatrix} \nabla_{(x_0,y_0)} f \\ -1 \end{bmatrix}$ is a vector perpendicular to the tangent plane to the surface at the point $(x_0, y_0, f(x_0,y_0))$.

Then the equation of the tangent plane must be:

$$\begin{bmatrix} \vec{\nabla}_{(x_0,y_0)} f \\ -1 \end{bmatrix} \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - f(x_0,y_0) \end{bmatrix} = 0$$

Or, more explicitely:

$$\boxed{z - f(x_0,y_0) = \frac{\partial f}{\partial x_0}\Big|_{(x_0,y_0)} (x - x_0) + \frac{\partial f}{\partial y_0}\Big|_{(x_0,y_0)} (y - y_0)}$$

Example: find the equation of the tangent plane to $z = x^2 + y$ at the point $(2,2)$.

1) $\vec{\nabla} f = \begin{bmatrix} 2x \\ 1 \end{bmatrix} \Rightarrow \vec{\nabla} f_{(2,2)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} x - 2 \\ y - 2 \\ z - 6 \end{bmatrix} = 0 \qquad \boxed{4(x-2) + (y-2) = z - 6}$$

EQUATION of THE TANGENT PLANE !!!

Exercises: Find the equation of the tangent plane at $(0,0)$ of the following functions:

1) $z = 1$

2) $z = e^{xy} + 3$

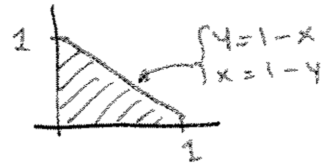
3) $z = \ln(x^2y^2+1)$

How about integration? Well, that's just a little trickier. Let me just mention the following powerful theorem, that we will use later on.

Theorem (Fubini): If I can split my integrand as a product of a function of x and a function of y, then I can integrate the integral "one piece at a time": i.e.

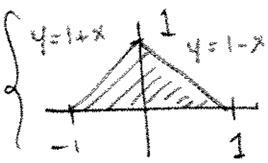
$$\int_{\text{Region}} f(x)\,g(y)\,dx\,dy = \int_{x\,\text{region}} f(x)\,dx \int_{y\,\text{region}} g(y)\,dy$$

$$= \int_{y\,\text{region}} g(y)\,dy \int_{x\,\text{region}} f(x)\,dx .$$

These "x region" and "y region" are horribly confusing, but we can make sense of it by looking at a couple of examples: 1) Find the area of 

$$\int_{\triangle} dx\wedge dy = \int_0^1 dy \int_0^{1-y} dx = \int_0^1 dy\,(1-y) = \left(y-\frac{y^2}{2}\right)\Big|_0^1 = \left(1-\frac{1}{2}\right)-(0) = \boxed{\frac{1}{2}}$$

Good!

2) Integrate $\int_R xy\,dx\wedge dy$ $\quad R = \Bigg\{$  $\Bigg\}$

CHOICE 1: integrate along the "x"'s first:

$$\int_0^1 y\,dy\int_{y-1}^{1-y} x\,dx = \int_0^1 y\left(\frac{x^2}{2}\right)\Big|_{y-1}^{1-y} = \int_0^1 y\,dy\left((1-y)^2-(y-1)^2\right) = \boxed{0}$$

CHOICE 2: integrate along the "y"'s first:

$$\int_{-1}^0 x\,dx\int_0^{1+x} y\,dy + \int_0^{+1} x\,dx\int_0^{1-x} y\,dy = \int_{-1}^0 x\,dx\left(\frac{y^2}{2}\right)\Big|_0^{1+x} + \int_0^1 x\,dx\left(\frac{y^2}{2}\right)\Big|_0^{1-x} =$$

$$= \int_{-1}^0\left(\frac{x^3}{2}+x^2+\frac{x}{2}\right)dx + \int_0^1\left(\frac{x^3}{2}-x^2+\frac{x}{2}\right)dx = \left(\frac{x^4}{8}+\frac{x^3}{3}+\frac{x^2}{4}\right)\Big|_{-1}^0 +$$

$$\left(\frac{x^4}{8}-\frac{x^3}{3}+\frac{x^2}{4}\right)\Big|_0^1 = \boxed{0}$$

Good! It checks, and it also checks with the intuitive fact that we are integrating over a region symmetric about the y-axis, a function that changes sign when I go from $x \rightsquigarrow -x$!!

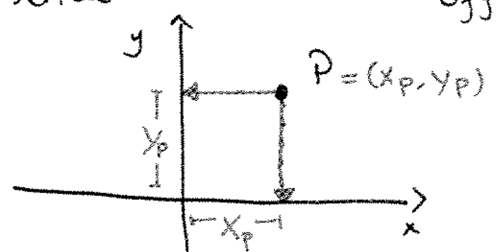
EXERCISES

1: Find the area of a circle using this method

2: $\int_R x\,dx\,dy$ $\quad R = \{$  $\}$

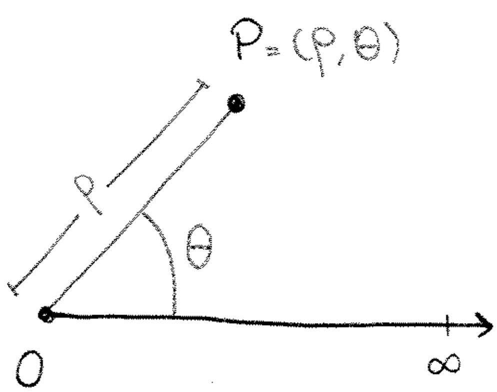
§2. Polar Coordinates

This is an alternative way of giving coordinates to the plane or to 3-dimensional space. But what does it mean, exactly, to give coordinates? It means to be able to identify a point in space given a set of numbers. In our usual (cartesian) system, we place orthogonal axes somewhere in space, and then we read off the projections from a given point onto these axes:



Given a pair of numbers (x_P, y_P) I can recover the point in space by just reversing the process. Now we are going to learn a method that, roughly, identifies a point by specifying its distance from the origin and its "direction".

Plane:



Fix a point on the plane and a half line departing at that point. We now identify a point P by giving: ρ = the distance from O to P

θ = the angle $P\hat{O}\infty$

Exercises: ① Give polar coordinates for the following (cartesian) points: $\to (0,0)$ $\to (-3,0)$

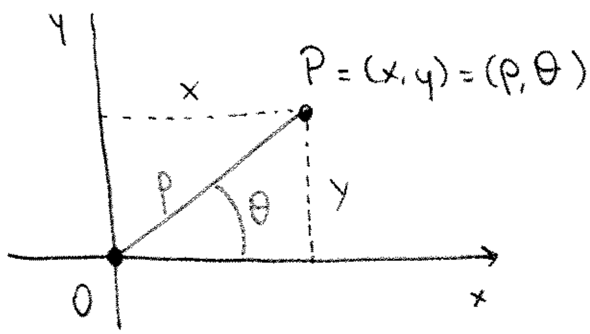
$\to (0,2)$ $\to (1,1)$

② Graph the following points:

$$(\rho,\theta) = (1,0)$$
$$= \left(\sqrt{2}, -\frac{\pi}{4}\right)$$
$$= (3, \pi/6)$$
$$= (8, \pi/2)$$

③ Find the cartesian coordinates for the above points.

There are obviously transformations that allow you to easily go from the cartesian to the polar system. You should have worked them out by doing the previous exercises. If not, here they are:



$$\boxed{\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned}}$$

POLAR TO CARTESIAN

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned}}$$

CARTESIAN TO POLAR

Now, exactly like we did in the cartesian world, we can study the equations of geometric loci expressed in polar coordinates. For some shapes the expression simplifies, for others it gets nasty.

Q1: what's the equation of a circle of radius 2 centered at the origin?

Q2: what loci are determined by the following equations:

1) $\rho = \theta$

2) $\rho\cos\theta = 4$

3) $\rho\sin\theta = 2$

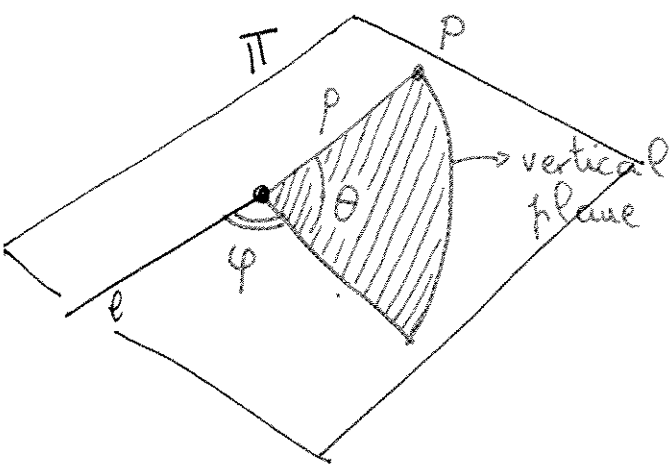
4) $\theta = \frac{\pi}{4}$

5) (harder) $\rho\cos(\theta - \pi/4) = \sqrt{2}$

6) (hard) $\rho = \sin\theta$

Q3: (for aficionados): work out the equation for a general line in polar coordinates. If you still haven't had enough, do it for circles as well!!

<u>3 dimensional space</u>



Here we fix a plane Π equipped with polar coordinates (ρ, φ) (I know, ... it'd be nicer to be consistent and stick with (φ, θ), but historically θ has been used for the "latitude" angle, so I have to stick with it).

Now a point P is identified with the following three numbers:

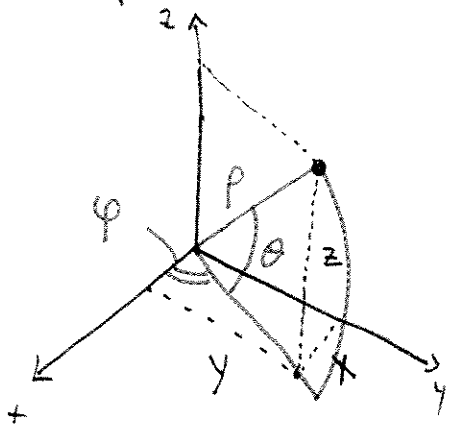
ρ = distance O to P (distance from the center of the earth)

θ = angle between OP and Π (latitude)

φ = angle between ℓ and the projection of OP onto Π (longitude)

3 dimensional polar coordinates are also called spherical coordinates.

In general we identify ℓ with the positive part of the x-axis. Then the transformations between cartesian and spherical are:



$$\boxed{\begin{aligned} x &= \rho\cos\theta\cos\varphi \\ y &= \rho\cos\theta\sin\varphi \\ z &= \rho\sin\theta \end{aligned}}$$

SPHERICAL to CARTESIAN.

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\frac{z}{\sqrt{x^2+y^2}} \\ \varphi &= \arctan\frac{y}{x} \end{aligned}}$$

CARTESIAN to SPHERICAL.

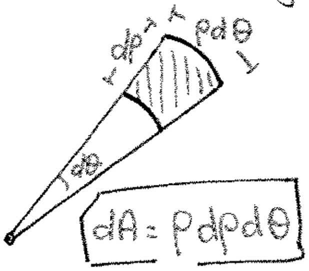
Q1: What's the spherical coordinates equation for a sphere of radius r centered at the origin?

Q2: What are the spherical coordinates equations for the

- x-axis?
- y-axis?
- z-axis?
- xy-plane?
- xz-plane?
- yz-plane?

Integration in polar coordinates

Sometimes it's very convenient to integrate in polar coordinates, as opposed to regular x-y coordinates, simply because the functions that we want to integrate, or the regions over which we are integrating them, may have a much better description in terms of polar coordinates. The rules of integration are

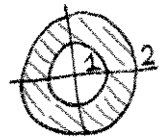


exactly the same, with only one caution: we have to replace the infinitesimal element of area $dx \wedge dy$ by $\rho d\rho \wedge d\theta$

EXAMPLE 1: Find the area of a circle of radius r using polar integration:

SOLUTION:
$$\int \rho d\rho d\theta = \int_0^r \rho d\rho \int_0^{2\pi} d\theta = 2\pi \int_0^r \rho d\rho = 2\pi \left(\frac{\rho^2}{2}\right)\Big|_0^r = \boxed{\pi r^2}$$

much simpler than when we tried in cartesian, uh?

EXAMPLE 2: Integrate $\int_R (x^2+y^2)\, dx \wedge dy$ $\quad R = \{$  $\}$

SOLUTION:
$$x^2+y^2 \rightsquigarrow \rho^2$$
$$dx \wedge dy \rightsquigarrow \rho d\rho \wedge d\theta$$
$$\Rightarrow \int_R \rho^3 d\rho \wedge d\theta = \int_1^2 \rho^3 d\rho \int_0^{2\pi} d\theta = 2\pi \left(\frac{\rho^4}{4}\right)\Big|_1^2 =$$
$$= 2\pi\left(4 - \frac{1}{4}\right) = \boxed{\frac{15}{2}\pi}$$

EXAMPLE 3: $\int_R x \, dx \wedge dy \qquad R = \Big\{$ 

$\Big\}$

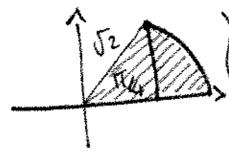
SOLUTION: $x \rightsquigarrow \rho \cos\theta$

$dx \wedge dy \rightsquigarrow \rho \, d\rho \wedge d\theta$

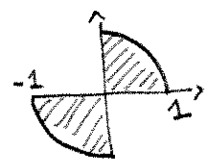
$$\Rightarrow \int_R \rho^2 \cos\theta \, d\rho \wedge d\theta = \int_0^2 \rho^2 d\rho \int_0^{\pi/4} \cos\theta \, d\theta = \left(\frac{\rho^3}{3}\right)\Big|_0^2 \cdot (\sin\theta)\Big|_0^{\pi/4} =$$

$$= \frac{8}{3} \cdot \left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{4\sqrt{2}}{3}}$$

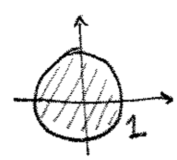
Exercises:

1) $\int_R y^2 \, dx \wedge dy \qquad R = \Big\{$ 

$\Big\}$

2) $\int_R (x^4 + 2x^2y^2 + y^4) \, dx \wedge dy \qquad R = \Big\{$ 

$\Big\}$

3) $\int_R \frac{1}{\sqrt{x^2+y^2}} \, dx \wedge dy \qquad R = \Big\{$ 

$\Big\}$

§3. Differential Forms:

A. FORMAL NONSENSE:

We introduce new formal gadgets dx, dy, dz. You can of course do this with as many coordinates as you want, but we will restrain to 3 for more clarity.
We also introduce an operation $\wedge$ (wedge) between these gadgets with the following bizarre properties:

(W1) $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$ — when you wedge 2 of the same you get 0

(W2) $dx \wedge dy = - dy \wedge dx$
$dx \wedge dz = - dz \wedge dx$
$dy \wedge dz = - dz \wedge dy$
— swapping 2 terms in $\wedge$ you get a minus sign!

Now, here are the differential forms:

0-FORMS: are just functions $\boxed{f(x,y,z)}$

1-FORMS: have "one" d☺ at a time: $\boxed{g_1(x,y,z)\,dx + g_2(x,y,z)\,dy + g_3(x,y,z)\,dz}$

2-FORMS: "two" d☺'s: $\boxed{h_1(x,y,z)\,dx\wedge dy + h_2(x,y,z)\,dx\wedge dz + h_3(x,y,z)\,dy\wedge dz}$

3-FORMS: "three" d☺'s: $\boxed{l(x,y,z)\,dx\wedge dy\wedge dz}$

Examples

0-FORM: $f(x,y,z) = x^2+y^2+z^2$

1-FORM: $y\,dx + x\,dy - z\,dz$

2-FORM: $dx \wedge dy$

3-FORM: $(45x^2 - y + \ln z)\,dx\wedge dy\wedge dz$

Question: why can't we have 4-forms if we restrain ourselves to 3-dimensional space?

8. THE DIFFERENTIAL

d is an operator that takes an n-form to an $(n+1)$-form. The way d works is pretty simple:

$$\boxed{d\left(f(x,y,z)\, d\text{☺}\wedge\dots\wedge d\star\right) = \frac{\partial f}{\partial x}\, dx\wedge d\text{☺}\wedge\dots\wedge d\star + \frac{\partial f}{\partial y}\, dy\wedge d\text{☺}\wedge\dots\wedge d\star + \frac{\partial f}{\partial z}\, dz\wedge d\text{☺}\wedge\dots\wedge d\star}$$

and then d is linear, i.e. $d(\text{sum}) = \text{sum}$ of the d's.

Examples:

0-Form to 1-form: $d\left(x^2+y^2-z\right) = 2x\,dx + 2y\,dy - dz$

1 Form to 2-Form: $d\left(x^2dx - (z+y)\,dy\right) = 2x\,\overset{0}{\cancel{dx\wedge dx}} + 0\,dy\wedge dx + 0\,dz\wedge dx$
$+ 0\,dx\wedge dy - 1\,\overset{0}{\cancel{dy\wedge dy}} - 1\,dz\wedge dy$

$= dy\wedge dz$

2FORM to 3-FORM: $d\left(x\,dx\wedge dy + y\,dx\wedge dz\right) = 1\cdot \overset{0}{\cancel{dx\wedge dx}}\wedge dy + 0\,dy\wedge dx\wedge dy +$
$+ 0\,dz\wedge dx\wedge dy + 0\,dx\wedge dx\wedge dz + 1\,dy\wedge dx\wedge dz + 0\,dz\wedge dx\wedge dz$

$= -\,dx\wedge dy\wedge dz$

Exercises: 1) $d\left(e^{xy} + 2z\right) =$

2) $d\left(e^y dx - z\,dz\right) =$

3) $d\left((x^2+y^2+z^2)\,dx\wedge dy\right) =$

4) $d\left(d\left(x^3 + 4y^4 - 5z^2\right)\right) =$

5) $d(d((x^3+y^3)\,dx - e^{xyz}\,dy + dz)) = ?$

Noticed something?

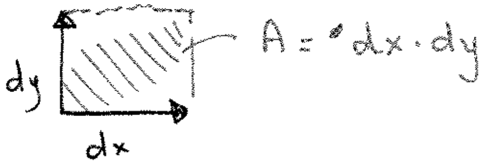
Yes, that is always true, and it's a fundamental thing about differential forms:

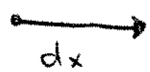
$$\boxed{\text{THEOREM: } \quad d \circ d = 0}$$

C. A LITTLE BIT of INTUITION (handwavy)

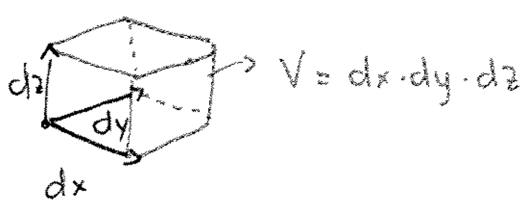
You can informally think of dx, dy, dz, as infinitesimal vectors pointing in the x, y, z directions (respectively).

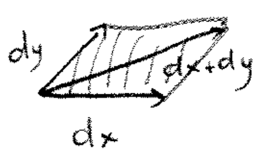
$\wedge$ translates to: "take the area (or volume) of the shape generated by the infinitesimal vectors". That's why:





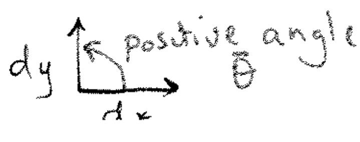
$dx \wedge dx = 0$, because it's a rectangle with no area!



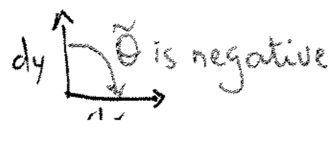


$V = 0 \Rightarrow dx \wedge dy \wedge (dx+dy) = 0$

The "minus sign" rule comes from the fact that $\wedge$ wants to keep track of the orientation of the shape generated



$\overset{\tilde{\theta}}{dx \wedge dy}$



$\overset{\tilde{\theta}}{dy \wedge dx}$

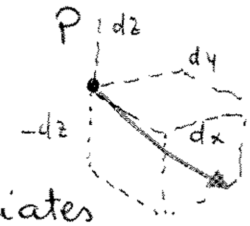
Now you can think of differential forms in the following way:

0-forms: just a regular function (or, in physics, a scalar field)

1-forms: a function that to any point in space associates a vector (i.e. a vector field)

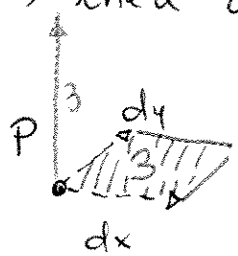
E.g. $x^2 dx + x dy - e^{2xyz} dz$

At the point $P = (1,2,0) \rightsquigarrow 1dx + 1dy - dz$



2-forms: a function that to any point in space associates a value and a little piece of plane. If we identify a plane with its normal vector, then again we recover a vector field:

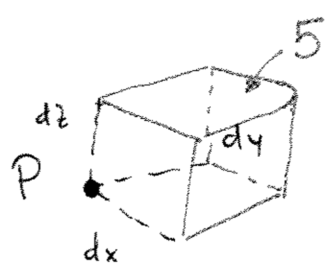
E.g. $3\, dx \wedge dy$



3-forms: they give values to little elements of volume:

E.g. $(3x^2 + y)\, dx \wedge dy \wedge dz$

At $P = (1,2,4)$



Differential forms are things that associate values to "infinitesimal" objects. Then it makes sense that these gadgets are really what we integrate, when we do integration!!

1 forms are integrated over curves, 2-forms over regions in the plane, 3-forms over regions of space. So, without knowing it, you have been integrating 1-forms for over one year now!!!

D. COOL APPLICATIONS 1 - DIFFERENTIAL OPERATORS IN PHYSICS

In the language of differential forms there is only one operator, d, and it's really easy to compute. But, if you identify scalar and vector fields with forms as shown before, you recover all the common differential operators in physics.

$$
\begin{tikzcd}
\{\text{0-FORMS}\} \arrow[r, "d"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "1"] & \{\text{1-FORMS}\} \arrow[r, "d"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "2"] & \{\text{2-FORMS}\} \arrow[r, "d"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "3"] & \{\text{3-FORMS}\} \arrow[d, leftrightarrow] \\
\left\{\begin{matrix}\text{SCALAR} \\ \text{FIELDS} \\ \text{(FUNCTIONS)}\end{matrix}\right\} \arrow[r, "\text{gradient}"] & \left\{\begin{matrix}\text{VECTOR} \\ \text{FIELDS}\end{matrix}\right\} \arrow[r, "\text{rotor}"] & \left\{\begin{matrix}\text{VECTOR} \\ \text{FIELDS}\end{matrix}\right\} \arrow[r, "\text{divergence}"] & \left\{\begin{matrix}\text{SCALAR} \\ \text{FIELDS}\end{matrix}\right\}
\end{tikzcd}
$$

Then, from the one and simple identity $d \circ d = 0$, we recover all the painful to remember differential identities of physics:

$$\text{rot}(\text{grad}(f)) = 0$$

$$\text{div}(\text{rot}(\vec{\phi})) = 0$$

Let me check for you the truth of ②. ① and ③ will be excellent exercises for who wants a good challange !!!

$$\phi_1\,dx + \phi_2\,dy + \phi_3\,dz \xrightarrow{\ d\ } \frac{\partial\phi_1}{\partial y}\,dy\wedge dx + \frac{\partial\phi_1}{\partial z}\,dz\wedge dx + \frac{\partial\phi_2}{\partial x}\,dx\wedge dy + \frac{\partial\phi_2}{\partial z}\,dz\wedge d$$

$$+ \frac{\partial\phi_3}{\partial x}\,dx\wedge dz + \frac{\partial\phi_3}{\partial y}\,dy\wedge dz =$$

$$= \left(\frac{\partial\phi_2}{\partial x} - \frac{\partial\phi_1}{\partial y}\right)dx\wedge dy + \left(\frac{\partial\phi_1}{\partial z} - \frac{\partial\phi_3}{\partial x}\right)dz\wedge dx +$$

$$+ \left(\frac{\partial\phi_3}{\partial y} - \frac{\partial\phi_2}{\partial z}\right)dy\wedge dz$$

$$\updownarrow$$

ANY VECTOR FIELD $\vec{\phi} = \begin{pmatrix} \phi_1(x,y,z) \\ \phi_2(x,y,z) \\ \phi_3(x,y,z) \end{pmatrix}$

$$\vec{\phi} \xrightarrow{\ \text{rot}\ } \begin{pmatrix} \left(\frac{\partial\phi_3}{\partial y} - \frac{\partial\phi_2}{\partial z}\right) \\ \left(\frac{\partial\phi_1}{\partial z} - \frac{\partial\phi_3}{\partial x}\right) \\ \left(\frac{\partial\phi_2}{\partial x} - \frac{\partial\phi_1}{\partial y}\right) \end{pmatrix} = \text{rot}\,\vec{\phi}$$

① and ② are <u>MUCH</u> easier to check, so don't be discouraged. Give it a try !!!

E. COOL APPLICATIONS 2: INTEGRATION

The cool thing is that dx is really $d(x)$, i.e. the operator d applied to the function x. This makes it extremely easy to change coordinates in integration. Let's see how we recover, in a completely straightforward way, the change of variables for polar integration:

From: $x = \rho \cos\theta$, $y = \rho \sin\theta$ we get: $dx = \cos\theta \, d\rho + (-\sin\theta)\rho \, d\theta$, $dy = \sin\theta \, d\rho + \cos\theta \, \rho \, d\theta$

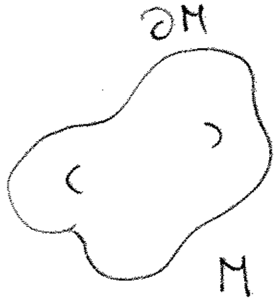
Then $$dx \wedge dy = (\cos\theta \, d\rho - \sin\theta \, \rho \, d\theta) \wedge (\sin\theta \, d\rho + \cos\theta \, \rho \, d\theta) =$$
$$= d\rho \wedge d\theta \left(\cos^2\theta \, \rho + \sin^2\theta \, \rho\right) = \rho \, d\rho \wedge d\theta \text{ !!!}$$

Exercise: find the spherical coordinate expression for the element of volume $dx \wedge dy \wedge dz$!

Final firework! There's a big theorem that unifies all the theorems about integration that you have started learning (FUNDAMENTAL THM of CALCULUS), and that you'll learn throughout calc III (GREEN'S THEOREM etc).

It goes as follows:

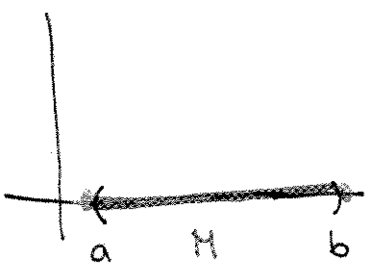
Stokes' thm



Let M be any bounded n-dimensional shape with boundary ∂M of dimension $n-1$. Let ω be any $(n-1)$-differential form. Then

$$\boxed{\int_{\partial M} \omega = \int_M d\omega}$$

FUNDAMENTAL THM of CALCULUS:



$\partial M = \{b\} - \{a\}$

In this case M is an interval $[a,b]$, with boundary $\{b\}-\{a\}$.

ω is a 0-form, i.e. a function F.

$d\omega$ is the 1-form $\frac{\partial F}{\partial x} dx$.

Then Stokes' theorem reads:

$$\int_{\{b\}-\{a\}} F = \int_a^b \frac{dF}{dx}\, dx$$

$$\|$$

$$F(b) - F(a)$$