Power Series

Math 1220 (Spring 2003)

A **power series** is a series that looks like:

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

where the a_n are real numbers and x is a variable. Notice that for power series, we always start off with a $k = 0$ term. This will be a minor annoyance.

Examples: A geometric power series is:

$$a + ax + ax^2 + ax^3 + ... = \frac{a}{1-x} \quad \text{if } |x| < 1$$

(and if $|x| \geq 1$ then the geometric power series diverges)

Definition: The **convergence set** of a power series is the set of values of x so that the resulting series converges. This is always be one of the following:

$$0, (-R, R), [-R, R), (-R, R], [-R, R] \text{ or } (-\infty, \infty)$$

We figure out R with the ratio test. Plug into the ratio test and solve for:

$$\lim_{k\to\infty} \frac{|a_{k+1} x^{k+1}|}{|a_k x^k|} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} |x| < 1$$

This gives an interval $(-R, R)$ fo values of x where:

$$\frac{1}{R} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

and the ratio test tells us **for sure** that the series converges in that interval and diverges in $(-\infty, -R) \cup (R, \infty)$. To figure out what happens at the endpoints $x = -R, R$ will require other tests.

Examples: (a) To get the convergence set for:

$$a + 2ax + 4ax^2 + 8ax^3 + ...$$

we first calculate:

$$\frac{1}{R} = \lim_{k\to\infty} \frac{2^{k+1}|a|}{2^k|a|} = 2$$

so $R = \frac{1}{2}$, and the series converges in $(-\frac{1}{2}, \frac{1}{2})$. If we plug in $x = \frac{1}{2}$, we get:

$$a + a + a + ...\text{which diverges}$$

and if we plug in $R = -\frac{1}{2}$, we get:

$$a - a + a - a + ...\text{which also diverges}$$

so the set of convergence is exactly:

$$(-\frac{1}{2}, \frac{1}{2})$$

(b) To get the convergence set for:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

(Note: We always adopt the convention that $0! = 1$) we solve:

$$\frac{1}{R} = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

so $R = \infty$, and the convergence set is all real numbers:

$$(-\infty, \infty)$$

(c) To get the convergence set for:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

we solve for:

$$\frac{1}{R} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

so $R = 1$. When we plug in $x = 1$, we get the alternating harmonic series (which converges!) and when we plug in $x = -1$, we get minus the harmonic series (which diverges!) so the convergence set is:

$$(-1, 1]$$