Power Functions and Log Functions

Math 1220 (Spring 2003)

You already know about the power functions x^n and x^r (for rational r). With our new technology we can now do even better. Fix a any real number.

$$x^a := \exp(a \ln(x))$$

(This looks superficially like a^x but it is a very different function of x!)

Properties of the Power Functions:

(o) $x^0 = 1, x^1 = x, x^{-1} = \frac{1}{x}$

(i) $x^{a+b} = x^a x^b$

(ii) $x^{-a} = \frac{1}{x^a}$

(iii) $x^{ab} = (x^a)^b$

(iv) If y is another variable, then $(xy)^a = x^a y^a$

(v) Also $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

So far, these are obvious, just reversing the roles of x and a. But:

(vi) $D_x(x^a) = a \cdot x^{a-1}$

(vii) $\int x^a dx = \frac{x^{a+1}}{a+1} + c$ (unless $a = -1$, of course)

Proof: (vi)

$$D_x(x^a) = D_x(\exp(a \cdot \ln(x)) = \exp(a \cdot \ln(x)) \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = a \cdot x^{a-1}$$

(vii) As usual, once we have $D_x(x^a)$, we can see the anti-derivative:

$$D_x(\frac{x^{a+1}}{a+1}) = \frac{1}{a+1} \cdot (a+1)x^a = x^a$$

So now for the first time we are allo wed to do funky things like:

Example: Evaluate $\int_1^2 x^\pi dx$

$$\int_1^2 x^\pi dx = \frac{x^{\pi+1}}{\pi+1}|_1^2 = \frac{2^{\pi+1}}{\pi+1} - \frac{1}{\pi+1}$$

and what the heck is $2^{\pi+1}$? Looking at the definitions, it is:

$$2^{\pi+1} = \exp((\pi+1) \cdot \ln(2)) \approx 17.65$$

In Between: Neither a power nor an exponential:

$$x^x := \exp(x \cdot \ln(x))$$

has derivative:

$$D_x(x^x) = D_x(\exp(x \cdot \ln(x)) = \exp(x \cdot \ln(x)) \cdot D_x(x \cdot \ln(x))$$

$$= x^x \cdot (\ln(x) + 1)$$

Example: Find the critical points of x^x on $(0, \infty)$ and classify them.

To find the critical points, we set:

$$D_x(x^x) = x^x \cdot (\ln(x) + 1) = 0$$

Since $x^x = \exp(x \ln(x))$ is always positive, this means that:

$$\ln(x) + 1 = 0 \text{ so } \ln(x) = -1 \text{ so } x = e^{-1} = \frac{1}{e}$$

is the only critical point. Taking second derivatives:

$$D_x^2(x^x) = D_x(x^x \cdot (\ln(x) + 1)) = (x^x \cdot (\ln(x) + 1))(\ln(x) + 1) + x^x \cdot \frac{1}{x}$$

$$= x^x((\ln(x) + 1)^2 + \frac{1}{x})$$

and this is always positive, so x^x is concave up, and the critical point is a local (and global) minimum. The minimum value of $y = x^x$ is:

$$y = (\frac{1}{e})^{\frac{1}{e}} \approx 0.6922$$

Most generally, we can take one function to the power of another function:

$$f(x)^{g(x)} = \exp(g(x) \ln(f(x))$$

and then the derivative of such an animal is (letting $u = g(x) \ln(f(x))$:

$$D_x(f(x)^{g(x)}) = D_x(\exp(u)) = \exp(u) \cdot u'$$

$$= f(x)^{g(x)} \cdot \left(g'(x) \cdot \ln(f(x)) + g(x) \cdot \frac{f'(x)}{f(x)} \right)$$

Example: Find $D_x(\sin(x)^{x^2})$. Following the above prescription:

$$D_x(\sin(x)^{x^2}) = \sin(x)^{x^2} \cdot (2x\ln(\sin(x)) + x^2\frac{\cos(x)}{\sin(x)})$$

Finally, let's revisit our old familiar log functions "with base a":

$$\log_a(x) := \frac{\ln(x)}{\ln(a)}$$

The more familiar definition says $\log_a(x)$ is the inverse function of a^x. Let's prove that this works:

$$\log_a(a^x) = \frac{\ln(a^x)}{\ln(a)} = \frac{\ln(\exp(x\ln(a)))}{\ln(a)} = \frac{x\ln(a)}{\ln(a)} = x$$

It works! Remember that log has the following properties:

$$\log_a(xy) = \log_a(x) + \log_a(y), \log_a(x^b) = b\log_a(x)$$

and

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

These properties are easy to see from the new definition!

Proof:

$$\log_a(xy) = \frac{\ln(xy)}{\ln(a)} = \frac{\ln(x)+\ln(y)}{\ln(a)} = \frac{\ln(x)}{\ln(a)} + \frac{\ln(y)}{\ln(a)} = \log_a(x) + \log_a(y)$$

$$\log_a(x^b) = \frac{\ln(x^b)}{\ln(a)} = \frac{b\ln(x)}{\ln(a)} = b\log_a(x)$$

$$\frac{\log_b(x)}{\log_b(a)} = \frac{\frac{\ln(x)}{\ln(b)}}{\frac{\ln(a)}{\ln(b)}} = \frac{\ln(x)}{\ln(a)} = \log_a(x)$$

And last but not least, some calculus:

$$D_x(\log_a(x)) = D_x(\frac{\ln(x)}{\ln(a)}) = \frac{1}{x\ln(a)}$$