Two More Tests for Convergence of Positive Series

Math 1220 (Spring 2003)

We will continue to assume that:

$$\sum_{k=1}^{\infty} a_k$$

is a positive series, which is to say that each $a_k \geq 0$.

Limit Comparison Test. If $\sum_{k=1}^{\infty} b_k$ is another positive series and $b_k \neq 0$ (in practice you will need to find this series for yourself!) and if

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L$$

then the two series are *comparable*, which means that:

(i) if $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ also converges

(ii) if $L > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum a_k$ also diverges.

Warning: If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ diverges, the test tell us nothing!

The test can allow us to conclude that messy series converge by comparing them with simpler series.

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + k - \sqrt{k}}$$

converges by comparison with the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ since:

$$\lim_{k \to \infty} \frac{1/(k^2 + k - \sqrt{k})}{1/(k^2)} = \lim_{k \to \infty} \frac{k^2}{(k^2 + k - \sqrt{k})} = 1$$

and we already know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges!

The Ratio Test. If

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = r$$

and if:

(i) $r < 1$, then the series converges.

(ii) $r > 1$, then the series diverges.

(iii) $r = 1$, then this test says nothing about the series.

Examples: (a)

$$\sum_{k=1}^{\infty} \frac{r^k}{k!}$$

converges for any $r \geq 0$, since

$$\lim_{k\to\infty} \frac{(r+1)^k/(k+1)!}{r^k/k!} = \lim_{k\to\infty} \frac{r}{k+1} = 0$$

(b) The ratio test doesn't tell us that the harmonic series diverges, since:

$$\lim_{k\to\infty} \frac{1/(k+1)}{1/k} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

(c) The ratio test doesn't tell us anything about **any** p-series, since:

$$\lim_{k\to\infty} \frac{1/(k+1)^p}{1/k^p} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^p = 1$$

and of course we know that some of these converge, and some don't!