Integrating by Parts

Math 1220 (Spring 2003)

It is a simple idea, but one of the most important of all calculus tools. When you see an integral that you recognize as being of the form:

$$\int u(x)v'(x)dx$$

for some functions $u(x)$ and $v(x)$, then the integral can be rewritten using:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Why does this work? Differentiate both sides to see!

$$\frac{d}{dx}(\int u(x)v'(x)dx) = u(x)v'(x) \text{ (by the fundamental theorem)}$$

and

$$\frac{d}{dx}(u(x)v(x) - \int v(x)u'(x)dx) = u'(x)v(x) + u(x)v'(x) - v(x)u'(x)$$

$$= u(x)v'(x) \text{ (by the product rule and the fundamental theorem)}$$

so they are the same!

Examples: (a)

$$\int xe^x dx$$

We set $u(x) = x$ and $v'(x) = e^x$ (so $v(x) = e^x$ also). Then we get:

$$\int xe^x dx = uv - \int vu'dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C$$

(b)

$$\int x\cos(x)dx$$

Here we set $u(x) = x$ and $v'(x) = \cos(x)$, so $v(x) = \sin(x)$. Then:

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + C$$

More Examples: (c)

$$\int \ln(x)dx$$

Set $u(x) = \ln(x)$ and $v'(x) = 1$, so $v(x) = x$. Then we get:

$$\int \ln(x)dx = x\ln(x) - \int \frac{1}{x} \cdot x dx = x\ln(x) - x + C$$

(Look familiar?)

(d)

$$\int \arcsin(x)dx$$

Set $u(x) = \arcsin(x)$ and $v'(x) = 1$, so $v(x) = x$. Then:

$$\int \arcsin(x)dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

We can soup up examples (a) and (b), to get "reduction" formulas by a repeated use of integration by parts:

Souped up Example (a)

$$\int x^n e^x dx$$

Set $u(x) = x^n$ and $v(x) = e^x$. Then:

$$\int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x$$

Now set $u(x) = x^{n-1}$ and $v(x) = e^x$:

$$\int x^{n-1} e^x dx = x^{n-1} e^x - \int (n-1) x^{n-2} e^x dx$$

and so on. This gives:

$$\int x^n e^x dx = x^n e^x - n x^{n-1} e^x + n(n-1) x^{n-2} e^x - ...$$

A final example: We can use integration by parts to integrate trig powers:

$$\int \sin^8(x)dx$$

Set $u = \sin^7(x)$ and $v = -\cos(x)$ (so that $v' = \sin(x)$). Then:

$$\int \sin^8(x)dx = -\sin^7(x)\cos(x) - \int 7\sin^6(x)\cos(x)(-\cos(x))dx$$

$$= -\sin^7(x)\cos(x) + 7\int \sin^6(x)(1 - \sin^2(x))$$

$$= -\sin^7(x)\cos(x) + 7\int \sin^6(x)dx - 7\int \sin^8(x)dx$$

Now bring the rightmost term over to the left and divide by 8. This gives(!)

$$\int \sin^8(x)dx = -\frac{1}{8}\sin^7(x)\cos(x) + \frac{7}{8}\int \sin^6(x)dx$$

But now we can treat the intgral of $\sin^6(x)$ in the same way to get:

$$\int \sin^6(x)dx = -\frac{1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\int \sin^4(x)dx$$

and we can continue on down to the bottom. The formula is messy, but for example if you want to do a definite integral like:

$$\int_0^{\frac{\pi}{2}} \sin^8(x)dx$$

you see that:

$$-\frac{1}{8}\sin^7(x)\cos(x)|_0^{\frac{\pi}{2}} = 0$$

so we can ignore this term (and all the others like it) and then:

$$\int_0^{\frac{\pi}{2}} \sin^8(x)dx = \frac{7}{8}\int_0^{\frac{\pi}{2}} \sin^6(x)dx = \left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\int_0^{\frac{\pi}{2}} \sin^4(x)dx = ...$$

$$... = \left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{2}} 1dx = \left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

and this is a lot easier than doing the same integral with the trig techniques!