Integrating by Partial Fractions

Math 1220 (Spring 2003)

If you are asked to perform an integral with a denominator that factors, then you need to use the method of partial fractions to perform the integral.

Example:

$$\int \frac{x}{x^2-1}dx = \int \frac{x}{(x-1)(x+1)}dx$$

The method of partial fractions says:

$$\frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

and we figure out A and B by:

$$A(x+1) + B(x-1) = x$$

$$(A+B)x + (A-B) = x$$

$$A+B = 1$$

$$A-B = 0$$

$$A = \frac{1}{2}, B = \frac{1}{2}$$

$$\int \frac{x}{x^2-1}dx = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx = \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + C$$

All such problems are done this way! Remember your partial fractions:

$$\frac{p(x)}{(x+a)(x^2+bx+c)} = \frac{A}{(x+a)} + \frac{Bx+C}{x^2+bx+c}$$

$$\frac{p(x)}{(x+a)(x+b)^2} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{Cx+D}{(x+b)^2}$$

We'll do a bunch of examples of these. Here are a couple:

Example:

$$\int \frac{x}{(x-1)(x^2+1)}dx = \int \frac{x}{(x-1)(x^2+1)}dx$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1)+(Bx+C)(x-1)=x$$
$$Ax^2+A+Bx^2-Bx+Cx-C=x$$
$$(A+B)x^2+(-B+C)x+(A-C)=x$$
$$A=-B, A=C, -B+C=-B-B=1$$
$$B=-\frac{1}{2}, A=\frac{1}{2}, C=\frac{1}{2}$$
$$\int \frac{x}{(x-1)(x^2+1)}dx=\frac{1}{2}\int \frac{1}{x-1}dx-\frac{1}{2}\int \frac{x-1}{x^2+1}dx$$
$$=\frac{1}{2}\int \frac{1}{x-1}dx-\frac{1}{2}\int \frac{x}{x^2+1}dx+\frac{1}{2}\int \frac{1}{x^2+1}dx$$
$$=\frac{1}{2}\ln(|x-1|)-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\tan^{-1}(x)$$

(Whew!)