The Natural Logarithm

Math 1220 (Spring 2003)

Here's a new function:

$$ln(x) := \int_1^x \frac{1}{t} dt$$

with domain $(0, \infty)$. Let's compare this function with some others:

$$\int_1^x 1dt = x - 1 \text{ (since } t \text{ is an antiderivative of } 1)$$

$$\int_1^x tdt = \frac{x^2}{2} - \frac{1}{2} \text{ (since } \tfrac{t^2}{2} \text{ is an antiderivative of } t)$$

in fact, whenever $n \neq -1$, then:

$$\int_1^x t^n dt = \frac{x^{n+1}}{n+1} - \frac{1}{n+1}$$

Notice, however, that if $n < 0$ then this only has domain equal to $\mathbf{R} - \{0\}$. So we can think of $ln(x)$ as a sort of analogue of this function.

OK. So now let's remember the 2nd fundamental theorem of calculus:

$$D_x(ln(x)) = D_x(\int_1^x \frac{1}{t} dt) = \frac{1}{x}$$

Or, in other words, $ln(x)$ is an anti-derivative of $\frac{1}{t}$. This is a mysterious function, but miraculously we can see that it has some interesting properties just coming from the definition! For example, from the chain rule:

$$D_x(ln(u)) = \frac{1}{u} D_x(u)$$

whenever $u = u(x)$ is a function of x. So for example:

$$D_x(ln(x^2)) = \frac{1}{x^2}(2x) = \frac{2}{x}$$

And, as a really interesting example:

$$D_x(ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}$$

so that $ln(-x)$ is *another* anti-derivative of $\frac{1}{x}$, but this one is only valid when $x < 0$, so putting them both together, we see that the indefinite integral:

$$\int \frac{1}{x} dx = ln(|x|) + c$$

So what does this have to do with the logarithm you've seen in pre-calc?

Properties of the Natural Logarithm:

(a) $ln(1) = 0$; $ln(x) < 0$ for $0 < x < 1$ and $ln(x) > 0$ for $1 < x$.

(b) $ln(ab) = ln(a) + ln(b)$ for positive numbers a and b.

(c) $ln(\frac{a}{b}) = ln(a) - ln(b)$ for positive numbers a and b.

(d) $ln(a^r) = rln(a)$ for rational numbers r and positive numbers a.

Proof: (a) comes from the fact that $\frac{1}{t} > 0$ when $t > 0$.:

$$\int_1^1 \frac{1}{t}dt = 0, \int_1^x \frac{1}{t}dt = \text{area under the graph if } x > 1$$

$$\int_1^x \frac{1}{t}dt = -\int_x^1 \frac{1}{t}dt = -\text{area under the graph if } x < 1$$

(b)

$$D_x(ln(ax)) = \frac{1}{ax}a = \frac{1}{x}$$

so $ln(ax)$ is an antiderivative of $\frac{1}{x}$. Since any two anti-derivatives of the same function differ by a constant:

$$ln(ax) = ln(x) + C$$

for some constant C. We can evaluate C by letting $x = 1$. This gives:

$$ln(a) = C$$

and thus:

$$ln(ax) = ln(a) + ln(x)$$

which is exactly what we wanted! (Set $x = b$)

(c) Notice first that:

$$0 = ln(1) = ln(b \cdot \frac{1}{b}) = ln(\frac{1}{b}) + ln(b)$$

so $ln(\frac{1}{b}) = -ln(b)$. Now use (ii):

$$ln(\frac{a}{b}) = ln(a \cdot \frac{1}{b}) = ln(a) + ln(\frac{1}{b}) = ln(a) - ln(b)$$

(d) Let's use the same trick we used in (ii). We have:

$$D_x(ln(x^r)) = \frac{1}{x^r}(rx^{r-1}) = \frac{r}{x}$$

but we also have:

$$D_x(rln(x)) = \frac{r}{x}$$

so that:

$$ln(x^r) = rln(x) + C$$

and what is C? Well, if we let $x = 1$, then we see that $C = 0$(!) So

$$ln(x^r) = rln(x)$$

for every x, which is what we want!

Final Remark: Since the natural log function satisfies:

$$D_x(ln(x)) = \frac{1}{x} > 0$$

and

$$D_x^2(ln(x)) = -\frac{1}{x^2} < 0$$

we see that it is everywhere increasing and concave down. Also:

$$\lim_{x\to\infty} ln(x) = \infty$$

and

$$\lim_{x\to 0^+} ln(x) = -\infty$$

so the range of $ln(x)$ is $(-\infty, \infty)$.