

More Power Series

Math 1220 (Spring 2003)

We can think of the power series

$$\sum_{k=0}^{\infty} a_k x^n$$

as a **function** with domain equal to the set of convergence.

Properties of the Power Series Function: Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^n$$

(i) $f(x)$ is a continuous function on its radius of convergence.

(ii) $f(x)$ is differentiable on the **interior** $(-R, R)$ of the convergence set and we can differentiate $f(x)$ term by term, giving:

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

which also converges on $(-R, R)$ (and possibly one or both of the endpoints).

(iii) $f(x)$ can be integrated term by term:

$$\int_0^x f(t) dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

for all x in the set $(-R, R)$, and this power series, too, converges on $(-R, R)$ and possibly one or both of the endpoints.

Examples: (a) The power series:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

satisfies $f'(x) = f(x)$ and $f(0) = 1$, and the convergence set is $(-\infty, \infty)$. So:

$$f(x) = e^x$$

and then, for example, $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots = e$.

(b) The power series:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

satisfies $f''(x) = -f(x)$, $f(0) = 1$, $f'(0) = 0$ with convergence set $(-\infty, \infty)$, so it is reasonable to conclude (and in fact true) that:

$$f(x) = \cos(x)$$

Similarly,

(c)

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \cosh(x)$$

(d)

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin(x)$$

(e)

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sinh(x)$$

For a different class of examples, start with:

$$1 - x + x^2 - x^3 + \dots = \frac{1}{1+x} \text{ with convergence set } (-1, 1)$$

and integrate:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \ln(1+x) \text{ with convergence set } (-1, 1]$$

which gives us the advertised limit of the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \dots = \ln(1+1) = \ln(2)$$

Or differentiate:

$$-1 + 2x - 3x^2 + \dots = -\frac{1}{(1+x)^2} \text{ with convergence set } (-1, 1)$$

We can do the same thing with:

$$1 - x^2 + x^4 - x^6 + \dots = \frac{1}{1+x^2} \text{ with convergence set } (-1, 1)$$

and then integrate to get:

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \tan^{-1}(x)$$

with convergence set $(-1, 1]$ again, and a **very** interesting limit:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \tan^{-1}(1) = \frac{\pi}{4}$$