More Improper Integrals

Math 1220 (Spring 2003)

By using limits, we can do integrals with integrands that blow up:

Blowing up at an endpoint: If $f(x)$ is continuous on (a, b) but

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ and/or } \lim_{x\to b^-} f(x) = \pm\infty$$

then we define the integral from a to b by taking limits of integrals:

$$\int_c^d f(x)dx$$

as $c \to a^+$ and $d \to b^-$.

Examples:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{1} - \lim_{a\to 0^+} 2\sqrt{a} = 2$$

On the other hand:

$$\int_0^1 \frac{1}{x}dx = \ln(1) - \lim_{a\to 0^+} \ln(a) = \infty$$

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{a\to 1^-} \sin^{-1}(a) - \lim_{a\to -1^+} \sin^{-1}(a) = \sin^{-1}(1) - \sin^{-1}(-1)$$
$$= \pi$$

(as you could have figured out looking at the circle!)

Blowing up at an interior point: If $f(x)$ is continuous on all of $[a, b]$ *except* for one point $a < c < b$, then we need to break up the integral:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Important Example: The integral:

$$\int_{-1}^1 \frac{1}{x^2}dx$$

looks like it can be done.

If you do it naively, you get:

$$\int_{-1}^{1} \frac{1}{x^2} dx = -\frac{1}{x}|_{-1}^{1} = -1 - (-\frac{1}{-1}) = -2$$

but a moment's reflection should tell you that this is completely ridiculous! The integrand is always positive, but the integral turns out negative?! The problem is, of course, the value 0. To do this integral, we'd need to split it:

$$\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx$$

but

$$-\frac{1}{x}|_{-1}^{0} = \lim_{a \to 0^-} -\frac{1}{a} - (-\frac{1}{-1}) = \infty$$

and likewise for the other, so **this integral diverges**.

On the other hand, we can happily do an integral like:

$$\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}} dx = \int_{-1}^{0} \frac{1}{x^{\frac{2}{3}}} dx + \int_{0}^{1} \frac{1}{x^{\frac{2}{3}}} dx =$$

$$\left(\lim_{a \to 0^-} 3a^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} \right) + \left(3(1)^{\frac{1}{3}} - \lim_{a \to 0^+} 3a^{\frac{1}{3}} \right) = 6$$

and this time, the naive answer is OK.

SO BEWARE!