More on Exponential Growth (and Decay)

Math 1220 (Spring 2003)

Recall that last time we noticed that if we start with a function:

$$y = f(t)$$

and if we are told that this function satisfies:

$$\frac{dy}{dt} = ry$$

then we know the function up to a constant:

$$y = e^c e^{rt}$$

Notice that if we plug in $t = 0$, we get:

$$f(0) = e^c$$

and $f(0)$ is usually written as y_0, so for that reason we write:

$$y = y_0 e^{rt}$$

Let's think about continuously compounding interest again. If $y = f(t)$ is the amount of money you have in the bank after t years, and if your interest (at rate r) is being compounded continuously, this means that at any moment the *rate of change* of your bank account is equal to $r \times y$. In other words:

$$\frac{dy}{dt} = ry$$

which means your bank balance satisfies the differential equation above and:

$$y = y_0 e^{rt}$$

And what is y_0? It is the amount of money you have at time 0. That is,

$$y_0 = P$$

and we have recovered the formula for continuously compounding interest!

Question: What other processes "compund continuously"?

Example 1: Populations. In the absence of any downward pressures, e.g. famine or war, we can assume that if $y(t)$ is the (large) population of a country, then:

$$\frac{dy}{dt} = ry$$

for some "rate" r, since the number of babies born should be (roughly) proportional to the number of people! So we know that:

$$y = y_0 e^{rt}$$

Normally you are not given r (this isn't a bank account!) but rather you are given the population at 2 different times and are asked to figure out r, and then use that to figure out the population at other times.

Sample: Suppose the population of a large country is:

(a) $200,000,000$ in 1950 and (b) $280,000,000$ in 1990.

Barring famine or war, what will it be in 2050? 2200? 3000?

Solution: Let us suppose that 1950 is time $t = 0$ (this simplifies matters!) Then we know from (a) that the population function satisfies:

$$y = (200,000,000)e^{r}t$$

but we don't know r. But plugging in (b) gives us:

$$280,000,000 = (200,000,000)e^{r\cdot 40}$$

which gives us:

$$r = \frac{1}{40}\ln(\frac{28}{20}) \approx 0.008412$$

and then the answers to our question are:

$$y = (200,000,000)e^{0.008412\cdot 100} \approx 463,800,000 \text{ in } 2050$$

$$y = (200,000,000)e^{0.008412\cdot 250} \approx 1,638,000,000 \text{ in } 2200$$

$$y = (200,000,000)e^{0.008412\cdot 1050} \approx 6,954,000,000 \text{ in } 3000$$

When people figured out in the 18th century that populations should grow like this, it freaked them out. Obviously it can't really grow like this forever!

Example 2. Radioactive decay. Some substances (like Uranium) decay at a rate proportional to the amount of the substance. That means that the amount of the substance at time t (which we'll denote by y) satisfies:

$$\frac{dy}{dt} = rt$$

where this time r is a *negative* number. Usually we want, or we are given, the *half-life* t_{half} of the substance, which is the amount of time it takes the substance to decrease to 1/2 its original time. That is, we want to solve:

$$\frac{1}{2}y_0 = y_0 e^{rt_{\text{half}}}$$

which means that the half-life satisfies:

$$\ln(\frac{1}{2}) = rt_{\text{half}}$$

so if we are given the half-life, we get the rate and vice versa.

Sample: Suppose the rate of decay of Carbon 14 is:

$$r = -0.000121 \ \ \text{yr}^{-1}$$

Then the half-life is:

$$t_{\text{half}} = \frac{1}{r} \cdot \ln(\frac{1}{2}) \approx 5730$$

Sample: If the half-life of an unstable isotope is 2 seconds, how much will remain of a 5 gram sample after 3.2 seconds?

From the half-life, we get r:

$$r = \frac{1}{t_{\text{half}}} \cdot \ln(\frac{1}{2}) = -.34657 \text{ sec}^{-1}$$

and then after 3.2 seconds, we'll have:

$$y = 5e^{r \cdot 3.2} = 5e^{-.34657 \cdot 3.2} \approx 1.65 \text{ grams}$$

Exponential functions also appear in music. The well-tempered scale won out in the 17th century as the framework for Western music. Sound is a wave, and musical notes sound, well, musical, because they are regular waves. The feature of the well-tempered scale, consisting of the notes:

$$C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B, C$$

is that the frequency of the waves doubles as you pass from one C to the next C and increases exponentially in even increments as you pass through each "half-step" of the scale. Since there are 12 half-steps in the scale, that means that if the frequency at (the lower) C is F_0 cycles per second, then the other frequencies are:

$$2^{\frac{1}{12}}F_0 \approx 1.059F_0 \text{ for } C\#$$

$$2^{\frac{2}{12}}F_0 \approx 1.122F_0 \text{ for } D$$

$$2^{\frac{3}{12}}F_0 \approx 1.189F_0 \text{ for } D\#$$

$$2^{\frac{4}{12}}F_0 \approx 1.256F_0 \text{ for } E$$

$$2^{\frac{5}{12}}F_0 \approx 1.335F_0 \text{ for } F$$

$$2^{\frac{6}{12}}F_0 \approx 1.414F_0 \text{ for } F\#$$

$$2^{\frac{7}{12}}F_0 \approx 1.498F_0 \text{ for } G$$

$$2^{\frac{8}{12}}F_0 \approx 1.587F_0 \text{ for } G\#$$

$$2^{\frac{9}{12}}F_0 \approx 1.682F_0 \text{ for } A$$

$$2^{\frac{10}{12}}F_0 \approx 1.782F_0 \text{ for } A\#$$

$$2^{\frac{11}{12}}F_0 \approx 1.888F_0 \text{ for } B$$

$$2^{\frac{12}{12}}F_0 = 2F_0 \text{ for the next } C$$

If you are a musician, then you know that perfect fifths and fourths have "harmonics" in common, which is why violinists (even without perfect pitch) can tune their instruments so accurately by ear. The reason for the harmonics is that the perfect fourth (C to F) represents a multiplication of the frequency by 1.335, which is very nearly $\frac{4}{3}$, and this nearness to a simple ratio is pleasing to the ear. Similarly, even more so, 1.498 is very near to $\frac{3}{2}$.