More Exponential Functions

Math 1220 (Spring 2003)

So far we have the natural log function:

$$\ln(x) := \int_1^x \frac{1}{t} dt \quad \text{Domain: } (0, \infty), \text{ Range: } (-\infty, \infty)$$

and the "natural" exponential function:

$$\exp(y) := ln^{-1}(x) \quad \text{Domain: } (-\infty, \infty), \text{ Range: } (0, \infty)$$

with the magical properties:

$$D_x(\ln(x)) = \frac{1}{x} \text{ and } D_x(\exp(x)) = \exp(x)$$

It is time to exp to "ordinary" exponential functions:

Let a be a positive real number. The "base a" exponential function is:

$$a^x := \exp(x \cdot \ln(a))$$

The domain of a^x is $(-\infty, \infty)$ and the range is $(0, \infty)$ (just like $\exp(x)$). Also:

Properties of the function a^x:

(o) $a^0 = 1, a^1 = a$ and $a^{-1} = \frac{1}{a}$,

(i) $a^{x+y} = a^x a^y$

(ii) $a^{-x} = \frac{1}{a^x}$ (so with (i) this gives $a^{x-y} = \frac{a^x}{a^y}$)

(iii) $a^{xy} = (a^x)^y$

(iv) If b is another positive number, then $(ab)^x = a^x b^x$

(v) Also $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

(vi) $D_x(a^x) = \ln(a) \cdot a^x$ (it is NOT a^{x-1}!!!!!)

(vii) $\int a^x dx = \frac{a^x}{\ln(a)} + c$

Proof:

(o) $a^0 = \exp(0) = \ln^{-1}(0) = 1$, $a^1 = \exp(\ln(a)) = \ln^{-1}(\ln(a)) = a$.

$$a^{-1} = \exp(-\ln(a)) = \exp(\ln(\frac{1}{a})) = \ln^{-1}(\ln(\frac{1}{a})) = \frac{1}{a}$$

(i) First of all, with a trick (substituting $\ln(\exp(x))$ for x) we get:

$$\exp(x+y) = \exp(\ln(\exp(x)) + \ln(\exp(y)))$$

$$= \exp(\ln(\exp(x)\cdot\exp(y))) = \exp(x)\cdot\exp(y)$$

using the property $\ln(a) + \ln(b) = \ln(ab)$ of the natural log. Now:

$$a^{x+y} = \exp((x+y)\ln(a)) = \exp(x\ln(a) + y\ln(a))$$

$$= \exp(x\ln(a))\cdot\exp(y\ln(a)) = a^x\cdot a^y$$

(ii) Using (o) and (i), we see that:

$$a^x \cdot a^{-x} = a^0 = 1$$

(iii)

$$(a^x)^y = \exp(y\ln a^x) = \exp(y\cdot x\ln(a)) = a^{xy}$$

using the property: $\ln(a^x) = x\cdot\ln(a)$ of the natural log.

(iv)

$$(ab)^x = \exp(x\ln(ab)) = \exp(x(\ln(a)+\ln(b))) = a^x b^x$$

(v)

$$\left(\frac{a}{b}\right)^x = (ab^{-1})^x = a^x(b^{-1})^x = a^x b^{-x} = \frac{a^x}{b^x}$$

(vi)

$$D_x(a^x) = D_x(\exp(x\ln(a)) = \exp(x\ln(a))\cdot\ln(a) = \ln(a)\cdot a^x$$

(vii) Since a (and $\ln(a)$) is a constant, we see that:

$$D_x(\frac{a^x}{\ln(a)}) = \frac{D_x(a^x)}{\ln(a)} = \frac{\ln(a)\cdot a^x}{\ln(a)} = a^x$$

which shows that $\frac{D_x(a^x)}{\ln(a)}$ is an anti-derivative of a^x.

A New Number: $\exp(x)$ itself is one of these new exponential functions. We define:

$$e := \ln^{-1}(1)$$

so that $\ln(e) = \ln(\ln^{-1}(1)) = 1$. Then:

$$\exp(x) = \exp(x\ln(e)) = e^x$$

so with this new number e as the base, our friend $\exp(x)$ is an exponential, and therefore satisfies all the properties above.

What sort of a number is e? Well, it is kind of like π in many ways. It has an infinite, non-repeating decimal expansion that starts:

$$e \approx 2.718281828459$$

Like π, it is defined very geometrically, as the number so that:

$$\int_1^e \frac{1}{t}dt = 1$$

(i.e. so that the area under the graph of $\frac{1}{t}$ should be 1).

Examples: (a) Find $D_x(a^{(x^2)})$. Use the chain rule (and let $u = x^2$):

$$D_x(a^{(x^2)}) = \ln(a)a^{(x^2)} \cdot 2x$$

(b) Find $D_x((a^x)^2)$. First simplify and then differentiate:

$$D_x((a^x)^2) = D_x(a^{2x}) = \ln(a)a^{2x} \cdot 2$$

or else go straight ahead with the chain rule (and let $u = a^x$):

$$D_x((a^x)^2) = 2a^x D_x(a^x) = 2a^x \cdot \ln(a)a^x = 2\ln(a)a^x \cdot a^x = 2\ln(a)a^{2x}$$

(c) Find $\int_1^2 2^x dx$. Use the anti-derivative:

$$\int_1^2 2^x dx = \frac{2^x}{\ln(2)}|_1^2 = \frac{4}{\ln(2)} - \frac{2}{\ln(2)} \approx 2.88539$$