The Basic Maclaurin Series

Math 1220 (Spring 2003)

Here are the basic Maclaurin Series from which you can build most any Maclaurin series you'd ever need by doing algebra or calculus.

1. The Maclaurin series of the exponential has convergence set $(-\infty, \infty)$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

2. The Maclaurin series of sine and cosine have convergence set $(-\infty, \infty)$,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

3. The Maclaurin series of cosh and sinh have convergence set $(-\infty, \infty)$,

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

4. The Maclaurin series of $\ln(1+x)$ has convergence set $(-1, 1]$,

$$\ln(1+x) = \int_0^x \frac{1}{1+x} dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

5. The Maclaurin series of $\tan^{-1}(x)$ has convergence set $(-1, 1]$,

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

There is one more important Maclaurin series you need to know:

Remember the **binomial theorem:**

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1+x$$

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$$\vdots$$

$$(1+x)^p = 1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+...+\frac{p!}{p!}x^p$$

We can see this as a Maclaurin series (or polynomial):

$$f(x) = (1+x)^p$$

$$f(0) = 0,\ f'(x) = p(1+x)^{p-1}$$

$$f'(0) = p,\ f''(x) = p(p-1)(1+x)^{p-2}$$

$$f''(0) = p(p-1),\ f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$$

$$\vdots$$

so $a_0 = 1, a_1 = p, a_2 = \frac{p(p-1)}{2!}, a_3 = \frac{p(p-1)(p-2)}{3!}$ and so on. If p is a positive integer, then this stops at $a_p = \frac{p!}{p!}$ because all future derivatives are zero. But if p isn't a positive integer, then this goes on forever, and produces the Maclaurin series for $(1+x)^p$.

6. The "binomial" Maclaurin series of $(1+x)^p$, for any p, is:

$$(1+x)^p = 1+px+\frac{p(p-1)}{2!}x^2+... = \sum_{k=0}^{\infty}\frac{p(p-1)...(p-k)}{k!}x^k$$

and this has a convergence set that always includes $(-1,1)$, but can be bigger!

Examples: (a) The Maclaurin series of $\frac{1}{1+x} = (1+x)^{-1}$ is, as we've seen:

$$1 - x + x^2 - x^3 + ... = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + ...$$

(b) The Maclaurin series of $(1+x)^{\frac{1}{2}} = \sqrt{1+x}$ is:

$$1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + ... = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - ...$$

(c) The Maclaurin series of $(1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$ is:

$$1 + (-\frac{1}{2})(-x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x^2)^3 + ...$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + ...$$

(d) The Maclaurin series of $\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1-x^2}}dx$ is:

$$x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{102}x^7 + ...$$

(the integral of the Maclaurin series in (c)).

(e) Here's an amusing one. The Maclaurin series of $(1-x)^{-2}$ is:

$$1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + ... =$$

$$1 + 2x + 3x^3 + 4x^3 + ...$$

It is also possible to see this one from the geometric series:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$$

and then differentiating:

$$(1-x)^{-2} = \frac{d}{dx}(1-x)^{-1} = 1 + 2x + 3x^2 + 4x^3 + ...$$