

Maclaurin Series

Math 1220 (Spring 2003)

If $f(x)$ is a function, and 0 is in its domain, then we ask:

Question: Is there a power series equals $f(x)$ in its convergence set?

Answer: Yes, for “reasonable” functions, and if there is one, we can find it by taking derivatives and evaluating them at $x = 0$. The series we get is called the **Maclaurin series** of $f(x)$.

In fact, suppose:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then:

$$f(0) = a_0 \text{ and } f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$f'(0) = a_1 \text{ and } f''(x) = \sum_{k=2}^{\infty} k(k-1) x^{k-2}$$

$$f''(0) = 2a_2 \text{ and } f'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2) x^{k-3}$$

\vdots

$$f^{(n)}(0) = n! a_n \text{ and } f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n) x^{k-n}$$

We can turn around the first column and conclude that:

$$a_0 = f(0), a_1 = f'(0), a_2 = \frac{f''(0)}{2!}, \dots, a_n = \frac{f^{(n)}(0)}{n!}$$

Example: If $f(x) = e^x$, then we know that $f^{(n)}(0) = e^0 = 1$, so:

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!}, \dots, a_n = \frac{1}{n!}$$

and then we reconfirm something we previously figured out:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

A Nasty Example: It turns out that **all** the derivatives $f^{(n)}(0)$ are 0 for

$$f(x) = e^{-\frac{1}{x^2}}$$

Thus, if there were Maclaurin series for this function, it would be:

$$\sum_{k=0}^{\infty} 0x^k = 0$$

But this function isn't the zero function! So we conclude that there is something wrong with the Maclaurin series for this function! Here's the problem:

Error Analysis: The Maclaurin partial sums are off by an error term:

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(x)}{n!}x^{n-1} + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

where c is some number in the interval $[-x, x]$. If:

$$\lim_{n \rightarrow \infty} \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \rightarrow 0$$

for every c in the interval $[-x, x]$, then the Maclaurin series does match the function and x is in the convergence set.

Remark: In all reasonable examples, this limit is 0. However, in the nasty example above, it turns out that the limit is never 0, which is why the Maclaurin series never works out to match $e^{-\frac{1}{x^2}}$ for any x (other than $x = 0$).

Example: Let's do the error analysis on $f(x) = e^x$ at $x = 1$.

($n = 0$)

$$e^1 = 1 + \frac{e^c}{1!} \cdot 1^1 \text{ for some } c \text{ in } [-1, 1]$$

and since e^1 is the largest possible error, this gives:

$$0 \leq e^1 - 1 \leq e^1$$

which tells us only that $1 \leq e^1$.

($n = 1$)

$$e^1 = 1 + \frac{1^1}{1!} + \frac{e^c}{2!} \cdot 1^2$$

gives

$$0 \leq e^1 - 2 \leq \frac{e^1}{2}$$

which tells us that

$$2 \leq e^1 \leq 4$$

($n = 2$)

$$e^1 = 1 + \frac{1^1}{1!} + \frac{1^2}{2!} + \frac{e^c}{3!} \cdot 1^3$$

gives

$$0 \leq e^1 - \frac{5}{2} \leq \frac{e^1}{6}$$

which tells us that

$$\frac{5}{2} \leq e^1 \leq 3$$

($n = 3$)

$$e^1 = 1 + \frac{1^1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{e^c}{4!} \cdot 1^4$$

gives

$$0 \leq e^1 - \frac{8}{3} \leq \frac{e^1}{24}$$

which tells us that:

$$\frac{8}{3} \leq e^1 \leq \frac{64}{23}$$

Another Example: Let's do an error analysis on $f(x) = \cos(x)$ at $x = \frac{\pi}{4}$.

($n = 0$)

$$\cos\left(\frac{\pi}{4}\right) = 1 + \frac{-\sin(c)\pi}{1! \cdot 4}$$

(here we'll just use that $|\sin(c)| \leq 1$ to get)

$$1 - \frac{\pi}{4} \leq \cos\left(\frac{\pi}{4}\right) \leq 1 + \frac{\pi}{4}$$

which is, again, not a very good error estimate!

($n = 1$)

$$\cos\left(\frac{\pi}{4}\right) = 1 + \frac{0}{1!} \frac{\pi}{4} + \frac{-\cos(c)}{2!} \left(\frac{\pi}{4}\right)^2$$

and this, amusingly, gives us a little better estimate:

$$1 - \frac{1}{2} \left(\frac{\pi}{4}\right)^2 \leq \cos\left(\frac{\pi}{4}\right) \leq 1 + \frac{1}{2} \left(\frac{\pi}{4}\right)^2$$

($n = 3$)

$$\cos\left(\frac{\pi}{4}\right) = 1 - \frac{1}{2!} \left(\frac{\pi}{4}\right)^2 + \frac{\cos(c)}{4!} \left(\frac{\pi}{4}\right)^4$$

gives a rather good estimate:

$$0.6757 \sim 1 - \frac{1}{2!} \left(\frac{\pi}{4}\right)^2 - \frac{1}{4!} \left(\frac{\pi}{4}\right)^4 \leq \cos\left(\frac{\pi}{4}\right) \leq 1 - \frac{1}{2!} \left(\frac{\pi}{4}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{4}\right)^4 \sim 0.7074$$

(the real answer is, of course $\sqrt{2}/2 \sim 0.7071$)