First Order Linear Differential Equations

Math 1220 (Spring 2003)

You've already seen how to solve differential equations of the form:

$$\frac{dy}{dt} = f(y)g(t)$$

Namely, you separate the variables and solve for y using:

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$

This is called the *change of variables* technique. We've just seen this used, for example, in solving:

$$\frac{dy}{dt} = ry$$

to get:

$$y = y_0 e^{rt}$$

but of course it would solve other things, like:

Example: To solve:

$$\frac{dy}{dt} = ty$$

We separate variables:

$$\int \frac{1}{y}dy = \int t dt$$

$$\ln(y) = \frac{t^2}{2} + C$$

$$y = y_0 e^{\frac{t^2}{2}}$$

But if we are given the differential equation:

$$\frac{dy}{dt} = ty + 1$$

then we cannot separate the variables! What to do? Well, in this case there is something to do! In fact, any differential equation of the form:

$$\frac{dy}{dt} = f(t)y + g(t)$$

is called a *first order linear differential equation* and here is how to solve it:

Step 1: Bring the *linear* $f(t)y$ term to the left, leaving $g(t)$ on the right:

$$\frac{dy}{dt} - f(t)y = g(t)$$

Step 2: Multiply both sides by the *integrating factor* $e^{\int -f(t)dt}$:

$$\frac{dy}{dt}e^{\int -f(t)dt} - f(t)ye^{\int -f(t)dt} = g(t)e^{\int -f(t)dt}$$

Step 3: Rewrite the left side, using the product rule:

$$\frac{d}{dt}(ye^{\int -f(t)dt}) = \frac{dy}{dt}e^{\int -f(t)dt} - f(t)ye^{\int -f(t)dt}$$

so

$$\frac{d}{dt}(ye^{\int -f(t)dt}) = g(t)e^{\int -f(t)dt}$$

Step 4: Integrate out both sides:

$$ye^{\int -f(t)dt} = \int g(t)e^{\int -f(t)dt}dt$$

Step 5: Multiply both sides by the inverse of the integrating factor:

$$y = e^{\int f(t)dt} \int g(t)e^{\int -f(t)dt}dt$$

and that's the answer!

The key point is Step 2. Multiplying both sides by the integrating factor makes the left side into a derivative, and this makes all the difference!

Example: Let's do our old example:

$$\frac{dy}{dt} = ty + 1$$

Let's go through the steps:

Step 1:

$$\frac{dy}{dt} - ty = 1$$

Integrating factor:

$$e^{\int -tdt} = e^{-\frac{t^2}{2}}$$

Steps 2 and 3:

$$\frac{d}{dt}(ye^{-\frac{t^2}{2}}) = \frac{dy}{dt}e^{-\frac{t^2}{2}} - tye^{-\frac{t^2}{2}} = e^{-\frac{t^2}{2}}$$

Step 4:

$$ye^{-\frac{t^2}{2}} = \int e^{-\frac{t^2}{2}} dt$$

Step 5:

$$y = e^{\frac{t^2}{2}} \int e^{-\frac{t^2}{2}} dt$$

Confession Time: The answer has to stay like this, because it is impossible to perform this improper integral! But let's check the answer:

$$\frac{dy}{dt} = \frac{d}{dt}(e^{\frac{t^2}{2}}) \int e^{-\frac{t^2}{2}} dt + e^{\frac{t^2}{2}} \frac{d}{dt}(\int e^{-\frac{t^2}{2}} dt)$$

$$= te^{\frac{t^2}{2}} \int e^{-\frac{t^2}{2}} dt + e^{\frac{t^2}{2}} e^{-\frac{t^2}{2}} = ty + 1$$

Whew! Such a complicated solution to such a simple differential equation!!

More Examples: Differential equations of the form:

$$\frac{dy}{dt} = k_1 y + k_2 \text{ (where } k_1 \text{ and } k_2 \text{ are constants)}$$

can be solved two ways:

By Separation of Variables:

$$\int \frac{1}{k_1 y + k_2} dy = \int 1 dt$$

$$\frac{1}{k_1} \ln(k_1 y + k_2) = t + c$$

$$k_1 y + k_2 = e^{k_1(t+c)}$$

$$y = \frac{e^{k_1(t+c)}}{k_1} - \frac{k_2}{k_1} = \frac{e^c}{k_1} e^{k_1 t} - \frac{k_2}{k_1} = Ce^{k_1 t} - \frac{k_2}{k_1}$$

By the Integrating Factor:

Step 1:

$$\frac{dy}{dt} - k_1 y = k_2$$

Integrating factor:

$$e^{\int -k_1 dt} = e^{-k_1 t}$$

Steps 2 and 3:

$$\frac{d}{dt}(ye^{-k_1 t}) = k_2 e^{-k_1 t}$$

Step 4:

$$ye^{-k_1 t} = k_2 \int e^{-k_1 t} dt = k_2(-\frac{1}{k_1} e^{-k_1 t} + c)$$

Step 5:

$$y = -\frac{k_2}{k_1} + c\frac{k_2}{k_1} e^{k_1 t} = Ce^{k_1 t} - \frac{k_2}{k_1}$$

Same answer! We can solve for C by plugging in $t = 0$:

$$C = y_0 + \frac{k_2}{k_1}$$

and we get:

$$y = (y_0 + \frac{k_2}{k_1})e^{k_1 t} - \frac{k_2}{k_1}$$

There are many examples of differential equations of this form:

Example 1: Gravity with air resistance. The velocity of an object falling through the air is given by the differential equation:

$$\frac{dv}{dt} = -av - g$$

where $g > 0$ is the acceleration due to gravity, and $a > 0$ is the drag coefficient coming from the air resistance. Thus the solution is:

$$v = (v_0 + \frac{g}{a})e^{-at} - \frac{g}{a}$$

and notice that as $t \to \infty$, this has a limit of $-\frac{g}{a}$. In other words, in the presence of air resistance, an object does not increase its speed indefinitely, but rather approaches its "terminal velocity" of $-\frac{g}{a}$.

Example 2: Mixtures. Start with 120 gallons of salt water containing 75 pounds of salt. Add salt water containing 1.2 pounds of salt per gallon at a rate of 2 gallons per minute, and allow salt water to flow out of the solution at the same rate. Assuming that the salt water is being constantly stirred, so that the salt water flowing out has the same concentration of salt as the overall mixture, how much salt will there be in the mixture after 1 hour?

The differential equation governing the amount of salt is:

$$\frac{ds}{dt} = -\frac{1}{60}s + 2.4 \text{ pounds/minute}$$

with initial condition $s_0 = 75$. From this we get:

$$s = (75 - (2.4)(60))e^{-\frac{t}{60}} + (2.4)(60) = 144 - 69e^{-\frac{t}{60}}$$

and then:

$$s(60) = 144 - 69e^{-1} \approx 118.62 \text{ pounds}$$

Notice that as $t \to \infty$, the amount of salt approaches 144 pounds, which is the amount of salt as 1.2 pounds per gallon. That is, the mixture approaches the concentration of the salt that is being added to it.

Example 3: The *current* $I(t)$ in a simple electrical circuit consisting of a resistor (with resistance R ohms), an inductor (with inductance L henrys) and a battery (supplying a voltage of $E(t)$ volts) satisfies the equation:

$$L\frac{dI}{dt} + RI = E(t)$$

This is known as Kirchhoff's Law. If we flip a switch in the circuit to "on" and assume constant voltage $E(t) = E$, then $I_0 = 0$ and:

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{E}{L}$$

has solution:

$$I(t) = (I_0 + \frac{\frac{E}{L}}{-\frac{R}{L}})e^{-\frac{R}{L}t} - \frac{\frac{E}{L}}{-\frac{R}{L}} = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t}$$

so the current approaches the ratio $\frac{E}{R}$ as $t \to \infty$.