L'Hôpital's Rule

Math 1220 (Spring 2003)

Here's the rule.

0/0 **L'Hôpital:** If $f(x)$ and $g(x)$ are functions and both:

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

and if $f'(x)$ and $g'(x)$ have a limit at a, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

unless the limits of $f'(x)$ and $g'(x)$ are again zero, in which case you repeat! (And this rule works also for upper and lower limits).

Examples:

$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = \lim_{x \to 0} \frac{-\sin(x)}{2x} = \lim_{x \to 0} \frac{-\cos(x)}{2} = -2$$

Why does it work? The idea is the following. There is a point c in any interval $[a, b)$ such that:

$$\frac{f(b)}{g(b)} = \frac{f(b) - f(a)}{b - a} \cdot \frac{b - a}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

If we let b approach a, then the right side approaches the limit

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ (if this limit exists)}$$

(by reasoning similar to the mean value theorem) and then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

also exists, and is the limit of the left side.

∞/∞ **L'Hôpital:** If $f(x)$ and $g(x)$ are functions and both:

$$\lim_{x\to c} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

and if $f'(x)$ and $g'(x)$ have (possibly infinite) limits, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

unless the limits of $f'(x)$ and $g'(x)$ are again infinite, in which case you repeat! (And this rule works also for upper and lower limits).

Example:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = ... = \lim_{x\to\infty} \frac{n(n-1)...(2)(1)}{e^x} = 0$$

The remaining forms of L'Hôpital all reduce to one of the first two:

$0 \cdot \infty$ **L'Hôpital:** If

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$

becomes the $0/0$ form of L'Hôpital.

$\infty - \infty$ **L'Hôpital:** If $f(x)$ and $g(x)$ are fractions and

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$

(or both are $-\infty$) then put the two over a common denominator.

Example:

$$\lim_{x\to 0^+} \left(\frac{1}{x} + \ln(x+1)\right) = \lim_{x\to 0^+} \left(\frac{1 + x\ln(x+1)}{x}\right) = \lim_{x\to 0^+} \left(\frac{x/(x+1) + \ln(x+1)}{1}\right)$$

$$= -\infty$$

1^∞ **L'Hôpital:** If

$$\lim_{x\to a} f(x) = 1 \text{ and } \lim_{x\to a} g(x) = \infty$$

then first figure out:

$$\ln(\lim_{x\to a} f(x)^{g(x)}) = \lim_{x\to a} (\ln(f(x)) \cdot g(x)) = y$$

using the $0 \cdot \infty$ version of the rule, and then:

$$\lim_{x\to a} f(x)^{g(x)} = e^y$$

Example:

$$\ln(\lim_{x\to 1} x^{\frac{1}{x-1}}) = \lim_{x\to 1} \left(\ln(x) \cdot \frac{1}{x-1}\right) = \lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1}{x} = 1$$

so

$$\lim_{x\to 1} x^{\frac{1}{x-1}} = e$$

Similarly, taking logs will give the 0^0 and ∞^0 forms of the rule.