Inverse Trig Functions

Math 1220 (Spring 2003)

Recall the trig functions and their derivatives:

$$\sin(x),\ D_x(\sin(x)) = \cos(x)$$

$$\cos(x),\ D_x(\cos(x)) = -\sin(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)},\ D_x(\tan(x)) = \sec^2(x)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)},\ D_x(\cot(x)) = -\csc^2(x)$$

$$\sec(x) = \frac{1}{\cos(x)}, D_x(\sec(x)) = \sec(x)\tan(x)$$

$$\csc(x) = \frac{1}{\sin(x)}, D_x(\csc(x)) = -\csc(x)\cot(x)$$

As we discussed earlier, we can now find inverse functions:

$$\sin^{-1}(y),\ \text{Domain:}(-1,1)\ \text{Range:}(-\frac{\pi}{2},\frac{\pi}{2})$$

$$\cos^{-1}(y),\ \text{Domain:}(-1,1)\ \text{Range:}(0,\pi)$$

$$\tan^{-1}(y),\ \text{Domain:}(-\infty,\infty)\ \text{Range:}(-\frac{\pi}{2},\frac{\pi}{2})$$

$$\sec^{-1}(y),\ \text{Domain:}(-\infty,-1)\cup(1,\infty)\ \text{Range:}(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$$

Recall that we get the domain and range of the inverse function $f^{-1}(y)$ by restricting to domains where $f(x)$ is increasing (or decreasing) so that the inverse function exists, and then swapping them. For example, $\sin(x)$ is increasing on $(-\frac{\pi}{2},\frac{\pi}{2})$, $\cos(x)$ is decreasing on $(0,\pi)$, and $\tan(x)$ is increasing on $(-\frac{\pi}{2},\frac{\pi}{2})$. $\mathrm{Sec}(x)$ is a little different. It is increasing and positive on $(0,\frac{\pi}{2})$ and increasing and negative on $(\frac{\pi}{2},\pi)$, so we can keep both these intervals.

Recall the magical formula for the derivative of the inverse function:

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ whenever } y = f(x), \text{ or equivalently } f^{-1}(y) = x$$

Examples: (i) Find $\sin^{-1}(\frac{1}{2})$ and $(\sin^{-1})'(\frac{1}{2})$.

$$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6} \text{ because } \sin(\frac{\pi}{6}) = \frac{1}{2} \text{ and } \tfrac{\pi}{6} \text{ is in the range}$$

$$(\sin^{-1})'(\frac{1}{2}) = \frac{1}{\sin'(\sin^{-1}(\frac{1}{2}))} = \frac{1}{\cos(\frac{\pi}{6})} = \frac{2}{\sqrt{3}}$$

(ii) Find $\tan^{-1}(-1)$ and $(\tan^{-1})'(-1)$.

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

$$(\tan^{-1})'(-1) = \frac{1}{\sec^2(-\frac{\pi}{4})} = \cos^2(-\frac{\pi}{4}) = \frac{1}{2}$$

(iii) Find $\sec^{-1}(-2)$ and $(\sec^{-1})'(-2)$

$$\sec^{-1}(-2) = \frac{2\pi}{3} \text{ because } \cos(\frac{2\pi}{3}) = -\frac{1}{2} \text{ and } \frac{2\pi}{3} \text{ is in the range}$$

$$(\sec^{-1})'(-2) = \frac{1}{\sec(\frac{2\pi}{3})\tan(\frac{2\pi}{3})} = (-\frac{1}{2})(-\frac{1}{\sqrt{3}}) = \frac{1}{2\sqrt{3}}$$

OK. Now we want formulas for the derivatives of the inverse trig functions. We will be considerably helped by the following geometric results:

$$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$$

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$

$$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$$

$$\tan(\sec^{-1}(x)) = \begin{cases} \sqrt{x^2-1} \text{ if } x > 1 \\ \\ -\sqrt{x^2-1} \text{ if } x < -1 \end{cases}$$

(see the triangles on page 354 of the text for a visual proof)

Notice that by using these results, you can often get exact answers for $\sin(\cos^{-1}(x))$ even when you cannot get exact answers for $\cos^{-1}(x)$.

Examples: (i)

$$\sin(\cos^{-1}(\frac{3}{5})) = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

(ii)

$$\sec(\tan^{-1}(4)) = \sqrt{1+16} = \sqrt{17}$$

Now for the Formulas:

$$(\sin^{-1})'(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos(\sin^{-1}(y)} = \frac{1}{\sqrt{1-y^2}}$$

$$(\cos^{-1})'(y) = \frac{1}{\cos'(x)} = \frac{1}{-\sin(\cos^{-1}(y))} = -\frac{1}{\sqrt{1-y^2}}$$

$$(\tan^{-1})'(y) = \frac{1}{\tan'(x)} = \frac{1}{\sec^2(\tan^{-1}(y))} = \frac{1}{1+y^2}$$

$$(\sec^{-1})'(y) = \frac{1}{\sec'(x)} = \frac{1}{\sec(\sec^{-1}(y))\tan(\sec^{-1}(y))}$$

$$= \begin{cases} \frac{1}{y\sqrt{y^2-1}} \text{ if } y > 1 \\ \\ -\frac{1}{y\sqrt{y^2-1}} \text{ if } y < -1 \end{cases} = \frac{1}{|y|\sqrt{y^2-1}}$$

Examples: Differentiate using the chain rule:

$$D_x(\sin^{-1}(e^x)) = D_x(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{e^x}{\sqrt{1-e^{2x}}}$$

(using the substitution $u = e^x$)

$$D_x(\tan^{-1}(x^a)) = D_x(\tan^{-1}(u)) = \frac{u'}{1+u^2} = \frac{ax^{a-1}}{1+x^{2a}}$$

(using the substitution $u = x^a$)

$$D_x(\sec^{-1}(a^x)) = \frac{\ln(a)a^x}{a^x\sqrt{a^{2x}-1}} = \frac{\ln(a)}{\sqrt{a^{2x}-1}}$$

More Examples: Some surprising improper integrals:

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x)|_0^{\frac{1}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

$$\int_1^{\sqrt{3}} \frac{1}{1+x^2}dx = \tan^{-1}(x)|_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Some Improper Integrals:

$$\int \frac{x}{1+x^4}dx =$$

Hmmm, looks like the derivative of $\tan^{-1}$. Let's try $u = x^2, du = 2xdx$

$$\int \frac{xdx}{1+x^4} = \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\tan^{-1}(u) = \frac{1}{2}\tan^{-1}(x^2)$$

A Word Problem: Suppose you are standing 4 miles from a rocket that takes off with a constant acceleration of 32 feet/sec^2 (so the crew feels 2Gs). What is the acceleration in the angle you see between the rocket and the horizon at time t?

The height of the rocket is given by the formula:

$$h(t) = 32\frac{t^2}{2} = 16t^2 \text{ (integrating the acceleration twice)}$$

so the angle you see (as a function of time t) is:

$$\tan^{-1}\left(\frac{16t^2}{4}\right) = \tan^{-1}\left(4t^2\right)$$

so to get the *acceleration* in this angle, we need to differentiate twice(!)

$$(\tan^{-1})'\left(4t^2\right) = \frac{8t}{1+16t^4}$$

$$(\tan^{-1})''\left(4t^2\right) = D_t(\frac{8t}{1+16t^4}) = \frac{8(1+16t^4) - 8t(64t^3)}{(1+16t^4)^2} = \frac{8(1-48t^3)}{(1+16t^4)^2}$$

measured in radians/sec^2.