Two Tests for Convergence of Positive Series

Math 1220 (Spring 2003)

A **positive** series is:

$$\sum_{k=1}^{\infty} a_k \text{ with every } a_k \geq 0$$

For the two tests below, we will assume we are given a positive series.

The Bounded Sum Test. If we somehow know that:

$$\sum_{k=1}^{\infty} a_k \leq U$$

then the series converges, and:

$$\sum_{k=1}^{\infty} a_k = L \leq U$$

Note: This is just hitting-the-ceiling in disguise!

Example:

$$1 + \frac{1}{2!} + \frac{1}{3!} + ... = \sum_{k=1}^{\infty} \frac{1}{k!}$$

converges by the bounded sum test because:

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}}$$

so:

$$\sum_{k=1}^{\infty} \frac{1}{k!} \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{4} + ... = 2$$

(it is a geometric series with $a = 1$ and $r = \frac{1}{2}$)

Notice that this doesn't tell us what the series converges to! In fact:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$$

as we shall see later.

The Integral Test. If there is a formula for the a_k

$$a_k = f(k)$$

and $f(x)$ is a continuous, non-increasing function with domain $[1, \infty)$, then:

$$\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} a_k$$

so

- if the integral is finite, then the series converges, and
- if the integral is infinite, then the series diverges.

Example. p-series: These are the (very important!) series:

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ... = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

Here the function is $f(x) = x^{-p}$ and we know from Chapter 9 that:

$$\int_1^{\infty} x^{-p} dx = \begin{cases} \frac{1}{p-1} \text{ if } p > 1 \\ \\ \infty \text{ if } p \leq 1 \end{cases}$$

so the p-series:

- converges if $p > 1$ and
- diverges if $p \leq 1$

Notice that when $p = 1$, this is the harmonic series again, which we see diverges for a second time. Notice also that once again, we do not know what these series converge to, when they do converge! In fact, it is very hard to see what they converge to!!

(3) (Limit Comparison Test.) Suppose $\sum a_k$ and $\sum b_k$ are positive series, and b_k is never zero. Suppose also that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

(i) if $L \geq 0$ and $\sum b_k$ converges then $\sum a_k$ converges

(ii) if $L > 0$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

(Warning: if $L = 0$ and $\sum b_k$ diverges, then we cannot conclude anything about $\sum a_k$.)

Examples:

(a) $\sum \frac{1}{ln(k) \cdot k^2}$ converges (apply the limit comparison test with $\sum \frac{1}{k^2}$.)

(b) $\sum \frac{ln(k)}{k}$ diverges (apply the limit comparison test with $\sum \frac{1}{k}$.)

(c) Comparing $\sum \frac{1}{ln(k) \cdot k}$ with $\sum \frac{1}{k}$ doesn't tell us anything about the series $\sum \frac{1}{ln(k) \cdot k}$. (But you can see that it diverges by the integral test!)

(4) (Ratio Test.) Suppose that $\sum a_k$ is a positive series, and suppose that $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = r$. Then:

(i) if $r < 1$, the series converges.

(ii) if $r > 1$, the series diverges.

(iii) if $r = 1$, the behavior of the series cannot be determined by this test.

Examples: (a) $\sum \frac{r^k}{k!}$ converges for any $r \geq 0$, since $\lim_{k \to \infty} \frac{r}{k+1} = 0$.

(b) $\sum \frac{k!}{k^k}$ converges. (You get $\frac{1}{e}$ as the limit of ratios!)

(c) The ratio test doesn't tell us that the harmonic series diverges, since $\lim_{k \to \infty} \frac{k}{k+1} = 1$.

(d) The ratio test doesn't tell us that the series $\sum \frac{1}{k^2}$ converges since $\lim_{k \to \infty} \frac{k^2}{(k+1)^2} = 1$.

An infinite **series** is the sum of the terms of an infinite sequence:

$$a_1 + a_2 + a_3 + a_4 + ...$$

The right way to think about this is as the sequence of the **partial sums:**

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, ...$$

and people like to give these partial sums their own names:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

etc

And there is some fancy notation so we don't have to write lots of symbols:

$$\{a_k\} \text{ stands for the } \textbf{sequence } a_1, a_2, a_3, a_4, ...$$

$$\sum_{k=1}^{\infty} a_k \text{ stands for the } \textbf{series } a_1 + a_2 + a_3 + a_4 + ...$$

and

$$S_n = \sum_{k=1}^{n} a_k \text{ stands for the } \textbf{partial sum } S_n = a_1 + a_2 + ... + a_n$$

Definition: When the sequence of partial sums of a series converges to L. That is, when:

$$\lim_{n \to \infty} S_n = L$$

then say that the series *converges to* L and we write:

$$\sum_{k=1}^{\infty} a_k = L$$

If the sequence of partial sums diverges, we say that the series diverges.

Example 1. Geometric Series: These are series of the form:

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + ...$$

The partial sums of this series are easy to calculate using a trick:

$$S_n - rS_n = (a + ar + ... + ar^{n-1}) - (ar + ar^2 + ... + ar^n) = a - ar^n$$

(assuming that $r \neq 1$) so that:

$$S_n = \frac{a - ar^n}{1 - r}$$

So:

If $|r| < 1$, then the series converges to:

$$\lim_{n\to\infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r}$$

But if $|r| \geq 1$, then the series diverges.

A No-Brainer Test for Divergence: If the **sequence** $\{a_k\}$ diverges, or

$$\lim_{k\to\infty} a_k \neq 0$$

then the **series** $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: Consider:

$$S_n - S_{n-1} = a_n$$

so in order for the sequence of S_n's to converge, it must be the case that the sequence of the a_n's converges to zero.

Warning! You can not turn this no-brainer into a test for **convergence!** There are plenty of sequences $\{a_k\}$ that converge to zero, but for which the series $\sum_{k=1}^{\infty} a_k$ do not converge!

Example 2. The Harmonic Series. The series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

diverges, even though obviously:

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

Here's why. We can group the terms of the harmonic series:

$$1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+...$$

in groupings of $1, 1, 2, 4, 8, 16, 32, ...$ Then each of the sums inside the parenthesis is $\geq \frac{1}{2}$, so the overall sum is greater than $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+...$ so it cannot have a finite limit!

Example 3. Collapsing Series. Here is one more series that does converge, and for which we can explicitly give the limit.

Suppose $\{a_k\}$ is a sequence with a formula that can be written:

$$a_k = f(k) - f(k-1)$$

Then the partial sums collapse upon each other:

$$S_1 = f(1) - f(0), S_2 = (f(1) - f(0)) + (f(2) - f(1)) = f(2) - f(0)$$

$$S_3 = (f(3) - f(2)) + (f(2) - f(1)) + (f(1) - f(0)) = f(3) - f(0)$$

and following the same pattern:

$$S_n = f(n) - f(0)$$

so that in this special example, if:

$$\lim_{k\to\infty} f(n) = 0$$

then the series $\sum_{k=1}^{\infty} a_k$ converges. In fact, it converges to:

$$\sum_{k=1}^{\infty} a_k = -f(0)$$

SubExample: Let:

$$a_k = \frac{1}{(k+1)(k+2)} = -\frac{1}{k+2} - \left(-\frac{1}{k+1}\right)$$

so in this subexample, we should let:

$$f(k) = -\frac{1}{k+2}$$

and then $a_k = f(k) - f(k-1)$, so we know immediately that:

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+... = \sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)} = -\left(-\frac{1}{0+2}\right) = \frac{1}{2}$$

which isn't immediately obvious!