Improper Integrals

Math 1220 (Spring 2003)

By using limits, we can do integrals with inifinite endpoints:

Infinite Endpoints:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx$$

Examples:

$$\int_0^\infty e^{-x}dx = \lim_{b\to\infty} -e^{-x}|_0^\infty = \lim_{b\to\infty}(-e^{-b}) - (-e^0) = 0 + 1 = 1$$

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\ln(x)|_1^\infty = \lim_{b\to\infty}\ln(b) - \ln(1) = \infty - 0 = \infty$$

But for $a > 1$:

$$\int_1^\infty \frac{1}{x^a}dx = \lim_{b\to\infty}\frac{x^{1-a}}{1-a}|_1^\infty = \lim_{b\to\infty}\frac{b^{1-a}}{1-a} - \frac{1}{1-a} = \frac{1}{a-1}$$

$$\int_{-\infty}^\infty xe^{-\frac{x^2}{2}}dx = \left(\lim_{b\to\infty} -e^{-\frac{b^2}{2}} - (-1)\right) + \left((-1) - \lim_{a\to-\infty} -e^{-\frac{a^2}{2}}\right)$$

$$= (0+1) + (-1-0) = 0$$

An Example from Probability: A function $f(x)$ satisfying:

- $f(x)$ has domain $(-\infty, \infty)$
- $f(x) \geq 0$ always
- $$\int_{-\infty}^\infty f(x)dx = 1$$

is called a probability density.

For this density, the probability that x is between a and b is:

$$\int_a^b f(x)dx$$

and the mean of the density is the "center of mass:"

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

while the variance (the amount it spreads out) is:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Example: The most important density is the Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

which we can't integrate right now(!) but we will see in Ch 16 that:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$$

which explains why this is a probability density. The mean is:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}} dx = 0$$

and then using integration by parts with $u = -x, v' = -xe^{-\frac{x^2}{2}}$ we get:

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = -xe^{-\frac{x^2}{2}} |_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

The integral on the right side is $\sqrt{2\pi}$, as we've been told. We calculate the other term using l'Hôpital's rule:

$$\lim_{x\to\infty} -xe^{-\frac{x^2}{2}} = \lim_{x\to\infty} -\frac{x}{e^{\frac{x^2}{2}}} = \lim_{x\to\infty} -\frac{1}{xe^{\frac{x^2}{2}}} = 0$$

and likewise for the $-\infty$ limit. So the variance is:

$$\sigma^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = 1$$