Hyperbolic Functions

Math 1220 (Spring 2003)

Hyperbolic functions are like trig functions, but built from exponentials:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Here's what they have in common with the trig functions:

$$\cosh^2(x) - \sinh^2(x) = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{2+2}{4} = 1$$

And, as far as calculus goes:

$$\frac{d}{dx}\cosh(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \frac{e^x - (-e^{-x})}{2} = \cosh(x)$$

so that:

$$\frac{d^2}{dx^2}\cosh(x) = \cosh(x) \text{ and } \frac{d^2}{dx^2}\sinh(x) = \sinh(x)$$

Once we have these, we define the other functions as in trig:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

and their derivatives can be calculated from the quotient rule, as in trig:

$$\frac{d}{dx}(\tanh(x)) = \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} = \operatorname{sech}^2(x)$$

$$\frac{d}{dx}(\coth(x)) = \frac{\sinh(x)\sinh(x) - \cosh(x)\cosh(x)}{\sinh^2(x)} = -\text{csch}^2(x)$$

$$\frac{d}{dx}(\text{sech}(x)) = -\frac{\sinh(x)}{\cosh^2(x)} = -\tanh(x)\text{sech}(x)$$

$$\frac{d}{dx}(\text{csch}(x)) = -\frac{\cosh(x)}{\sinh^2(x)} = -\coth(x)\text{csch}(x)$$

Examples: (a) Find $D_x(\tanh(x^2))$. By the chain rule:

$$D_x(\tanh(x^2)) = \text{sech}^2(x^2) \cdot 2x$$

(b) Find $\int \tanh(x)dx$. Substituting $u = \cosh(x), du = \sinh(x)dx$:

$$\int \tanh(x)dx = \int \frac{\sinh(x)}{\cosh(x)}dx = \int \frac{du}{u} = \ln(u) + C = \ln(\cosh(x)) + C$$

Now for the inverses! To find $\cosh^{-1}(y)$ we solve for x:

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$0 = e^x - 2y + e^{-x}$$

Multiply through by e^x:

$$0 = e^{2x} - 2ye^x + 1$$

and use the quadratic formula (which gives $\pm$, but $+$ is the correct choice)

$$e^x = \frac{2y + \sqrt{4y^2 - 4}}{2} = y + \sqrt{y^2 - 1}$$

So

$$\cosh^{-1}(y) = x = \ln(e^x) = \ln\left(y + \sqrt{y^2 - 1}\right)$$

and from this, we get:

$$\frac{d}{dy}(\cosh^{-1}(y)) = \frac{d}{dy}(\ln\left(y + \sqrt{y^2 - 1}\right)) = \frac{1}{y + \sqrt{y^2 - 1}}(1 + \frac{y}{\sqrt{y^2 - 1}})$$

$$= \frac{1}{y + \sqrt{y^2 - 1}}\left(\frac{\sqrt{y^2 - 1} + y}{\sqrt{y^2 - 1}}\right) = \frac{1}{\sqrt{y^2 - 1}}$$

Similarly(!) one can obtain the following derivatives:

$$\frac{d}{dy}(\sinh^{-1}(y)) = \frac{1}{\sqrt{y^2+1}}$$

$$\frac{d}{dy}(\tanh^{-1}(y)) = \frac{1}{1-y^2}$$

$$\frac{d}{dy}(\text{sech}^{-1}(y)) = -\frac{1}{x\sqrt{1-x^2}}$$

Examples: Find suitable domains and ranges of:

$$\cosh^{-1}(y)$$

Since $\cosh(x)$ is increasing on $(0, \infty)$ with range $(1, \infty)$, we get:

Domain:$(1, \infty)$ Range: $(-\infty, \infty)$

Find suitable domains and ranges of:

$$\sinh^{-1}(y)$$

Here $\sinh(x)$ is everywhere increasing, with range $(-\infty, \infty)$, so:

Domain: $(-\infty, \infty)$ and Range: $(-\infty, \infty)$

Examples: Find

$$\int \frac{e^x}{\sqrt{e^{2x}+1}} dx$$

Do the substitution $u = e^x, du = e^x dx$ to get:

$$\int \frac{e^x}{\sqrt{e^{2x}+1}} dx = \int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1}(u) + C = \sinh^{-1}(e^x) + C$$