Introducing The Exponential Function

Math 1220 (Spring 2003)

Notice that the natural logarithm function $ln(x)$ satisfies:

(a) the domain of $ln(x)$ is $(0, \infty)$

(b) the range of $ln(x)$ is $\mathbf{R} = (-\infty, \infty)$ (think about limits)

(c) $ln(x)$ is a strictly increasing function (because $D_x(ln(x)) > 0$)

A function $f(x)$ that satisfies (c) (or else is strictly decreasing) always has an *inverse function*, which mathematicians always write as $f^{-1}(y)$. The inverse function has the following properties:

(a) the domain of $f^{-1}(y)$ is the range of $f(x)$

(b) the range of $f^{-1}(y)$ is the domain of $f(x)$

(c) the graph of $f^{-1}(y)$ is the reflection of the graph of $f(x)$ across $y = x$

(d) the inverse function "undoes" $f(x)$, and vice versa. That is:

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

Notice that $f^{-1}(y)$ is NOT the same as $\frac{1}{f(y)}$. It is extremely unfortunate, but a historical fact, that this notation is a bit confusing. Sorry!

Using the chain rule, we can get another interesting property of the inverse function. Let $y = f(x)$. Then:

$$1 = D_x(x) = D_x(f^{-1}(f(x))) = D_y(f^{-1}(y)) \cdot D_x(f(x))$$

(e) As a result of the above calculation: $D_y(f^{-1}(y)) = \frac{1}{D_x(f(x))}$

Example: The inverse of the function $f(x) = x^3$ is the function $f^{-1}(y) = y^{\frac{1}{3}}$. The domain and range of both functions is $(-\infty, \infty)$ and:

$$D_y(y^{\frac{1}{3}}) = \frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{3}(x^3)^{-\frac{2}{3}} = \frac{1}{3x^2} = \frac{1}{D_x(x^3)}$$

(remember that we can substitute $y = x^3$ or $x = y^{\frac{1}{3}}$ at any time!)

Another Example: $f(x) = x^2$ does not have an inverse everywhere, but the two functions:

$$\sqrt{x} \text{ and } -\sqrt{x}$$

are inverses to $f(x)$ when restricted to the domains $[0, \infty)$ and $(-\infty, 0]$, respectively.

Another Example: Sometimes we can find the inverse by solving for x:

$$f(x) = \frac{x}{1+x}$$

has an inverse that we can calculate! Set $y = \frac{x}{1+x}$. Then:

$$y(1+x) = x; \quad y + yx = x; \; y = x - yx; \; y = x(1-y); \; x = \frac{y}{1-y}$$

so that:

$$f^{-1}(y) = \frac{y}{1-y}$$

So the domain of $f(x)$ and the range of $f^{-1}(y)$ are $(-\infty, -1) \cup (-1, \infty)$ and the domain of $f^{-1}(y)$ and the range of $f(x)$ are $(-\infty, 1) \cup (1, \infty)$.

Let's check a couple of things. First, let's check:

$$f^{-1}(f(x)) = \frac{f(x)}{1-f(x)} = \frac{\frac{x}{x+1}}{1-\frac{x}{x+1}} = \frac{\frac{x}{x+1}}{\frac{1}{x+1}} = x$$

$$f(f^{-1}(y)) = \frac{f^{-1}(y)}{1+f^{-1}(y)} = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y$$

Check!! Now let's check the derivatives:

$$D_x(f(x)) = \frac{(1+x)(1)-(1)(x)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$D_y(f^{-1}(y)) = \frac{(1-y)(1)-y(-1)}{(1-y)^2} = \frac{1}{(1-y)^2}$$

and if we plug in $y = \frac{x}{1+x}$, we will get (try it!):

$$D_y(f^{-1}(y)) = \frac{1}{D_x(f(x))}$$

Yet another example! The inverse of $\sin(x)$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) is:

$$arcsin(y) := sin^{-1}(y)$$

This is (for now) a mysterious function, but(!) we can say this:

$$D_y(arcsin(y)) = \frac{1}{D_x(sin(x))} = \frac{1}{cos(x)} = \frac{1}{\sqrt{1-sin(x)^2}} = \frac{1}{\sqrt{1-y^2}}$$

or, in other words, $\arcsin(x)$ + c is the anti-derivative of $\frac{1}{\sqrt{1-x^2}}$.

Here's a new function (called the "natural" exponential function):

$$exp(y) := ln^{-1}(y)$$

By what we said above, we know that:

(a) the domain of $exp(y)$ is $(-\infty, \infty)$

(b) the range of $exp(y)$ is $(0, \infty)$

and the truly amazing feature of the exponential function is:

$$D_y(exp(y)) = \frac{1}{D_x(ln(x))} = \frac{1}{\frac{1}{x}} = x = exp(y)$$

(again, we can substitute $y = ln(x)$ or $x = exp(y)$ anytime!)

In other words, the the exponential function is equal to its own derivative! This, as we shall see, makes the exponential function an extremely important function for calculus and in solving differential equations.