Thinking about Exponential Growth

Math 1220 (Spring 2003)

Here are our new functions and their derivatives and anti-derivatives:

$$\ln(x) = \int_1^x \frac{1}{t}dt;\ D_x(\ln(u)) = \frac{u'}{u};\int \ln(x)dx = x\ln(x) - x + c$$

$$\exp(x) = \ln^{-1}(x);\ D_x(\exp(u)) = \exp(u)\cdot u';\int \exp(x)dx = \exp(x) + c$$

$$a^x = \exp(x\ln(a));\ D_x(a^u) = \ln(a)\cdot a^u \cdot u';\int a^x dx = \frac{a^x}{\ln(x)} + c$$

$$x^a = \exp(a\ln(x));\ D_x(u^a) = a\cdot u^{a-1}\cdot u';\int x^a dx = \frac{x^{a+1}}{a+1} + c$$

Try graphing:

$$2^x, e^x, 3^x, 4^x$$

on your graphing calculator. They all look quite similar.

Now try graphing:

$$x^2, x^e, x^3, x^4$$

They look not too different from the exponential functions. But they are! It is not that easy to see this visually, but if we compose these functions with ln, then the differences become apparent. For example, compare:

$$\ln(e^x) = x, \ln(x^2), \ln(x^3), \ln(x^4)$$

Quite different! We'll evaluate limits later on to see:

$$y = e^x \text{ grows (much) faster than any power function } y = x^n$$

and

$$y = \ln(x) \text{ grows (much) more slowly than any power function } y = x^{\frac{1}{n}}$$

While we're on the subject of exponential growth, let's think about money.

Remember that if put P dollars in a bank account that gets $r\%$ per year, then after t years you will have:

- $A(1+r)^t$ dollars if interest is compounded yearly.
- $P(1+\frac{r}{12})^{12t}$ dollars if interest is compounded monthly.
- $P(1+\frac{r}{365})^{365t}$ dollars if interest is compounded daily.

But now what about if interest is compounded every hour, or every minute, or every second, or...

Definition: A *continuously* compounded interest rate gives:

$$P \lim_{n\to\infty}(1+\frac{r}{n})^{nt}$$

dollars after t years.

And what is this? Is it infinite? Actually, no. If we notice:

$$(1+\frac{r}{n})^{nt} = \left((1+\frac{r}{n})^{\frac{n}{r}}\right)^{rt}$$

then

$$P \lim_{n\to\infty}(1+\frac{r}{n})^{nt} = P \lim_{n\to\infty}\left((1+\frac{r}{n})^{\frac{n}{r}}\right)^{rt} = P\left(\lim_{n\to\infty}(1+\frac{r}{n})^{\frac{n}{r}}\right)^{rt}$$

(we can bring the limit inside the power because taking powers is continuous)

So we need to figure out (setting $h = \frac{r}{n}$):

$$\lim_{n\to\infty}(1+\frac{r}{n})^{\frac{n}{r}} = \lim_{h\to 0^+}(1+h)^{\frac{1}{h}}$$

In fact, I claim that this limit is our new number e. Here's why:

Let $f(x) = \ln(x)$, so that $f'(x) = \frac{1}{x}$ and in particular, $1 = f'(1)$. Now from the definition of the derivative, we therefore have:

$$1 = f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}$$
$$= \lim_{h\to 0}\left(\ln(1+h)\right)\frac{1}{h} = \lim_{h\to 0}\ln\left((1+h)^{\frac{1}{h}}\right)$$

Now we can take exp of both sides, and because exp is continuous:

$$e = \exp(1) = \exp(\lim_{h\to 0}\ln\left((1+h)^{\frac{1}{h}}\right)) = \lim_{h\to 0}\exp\ln\left((1+h)^{\frac{1}{h}}\right)$$
$$= \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

And this is very far from being infinite!

Example: Put \$100 in the bank for 10 years at 8%

(a) If you compound yearly, you'll have: \$215.89

(b) If you compound daily, you'll have: \$222.53

(c) If you compound continuously, you'll have:

$$Pe^{rt} = 100e^{(.08)(10)} = \$222.55$$

Yup. Only 2 cents more than daily!

The formula for interested compounded continuously:

$$y = Pe^{rt}$$

is an example of a function "with exponential growth." This arises naturally when we consider the following *differential equation* on a function $y = f(t)$.

Problem: Solve for $y = f(t)$ if y satisfies:

$$\frac{dy}{dt} = ry$$

Solution: Separate variables and integrate:

$$\int \frac{dy}{y} = \int r dt$$

$$\ln(y) = rt + c; \;\; y = \exp(\ln(y)) = \exp(rt + c) = e^c e^{rt}$$

so setting $P = e^c$ gives us back the formula for compound interest!

In general, we can solve for e^c if we are given one value of $f(t)$. This is called the "initial condition" for the differential equation.

Example: Solve for $y = f(t)$ with initial condition $f(5) = 7$ and diff eqn:

$$\frac{dy}{dt} = 2y$$

Solution: We just solved:

$$y = e^c e^{2t}, \text{ so plugging in } f(5) = 7 \text{ gives } 7 = e^c e^{10}$$

and $e^c = e^{10}/7$. So the function is:

$$f(t) = \frac{e^{10}}{7} e^{2t}$$