

Some Clever Substitutions

Math 1220 (Spring 2003)

Here are some very clever substitutions that I want to share with you.

When you see:

$$\sqrt[n]{ax + b}$$

in the integrand, you should always think to do the substitution:

$$u = \sqrt[n]{ax + b}$$

(OK, so that one wasn't brain surgery)

Example:

$$\int \frac{dx}{x + \sqrt{x}}$$

You see \sqrt{x} , you think to substitute $u = \sqrt{x}$. We'll work it out!

Other examples:

$$\int x^2 \sqrt{x+1} dx$$

One subtle point. You do $u = \sqrt{x+1}$ and then $u^2 = x+1$ so $2udu = dx$.

When you see:

$$\sqrt{a^2 - x^2} \text{ or } \sqrt{a^2 + x^2}$$

in the integrand, then you do the substitution:

$$x = a \sin(u), \text{ or } x = a \tan(u)$$

The point is that trig identities then streamline the integral. This works best with the restriction $-\frac{\pi}{2} < x < \frac{\pi}{2}$ which we will always assume.

Example:

$$\int \sqrt{a^2 - x^2} dx$$

then $x = a \sin(u)$, so

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2(u)} = \sqrt{a^2(1 - \sin^2(u))} = \sqrt{a^2 \cos^2(u)} = a \cos(u)$$

and when you put this together with

$$dx = a \cos(u) du$$

you get:

$$\int \sqrt{a^2 - x^2} dx = \int (a \cos(u))(a \cos(u)) du$$

which we know how to solve! We'll finish this one in class.

Similarly,

When you see:

$$\sqrt{x^2 - a^2}$$

you should think to do the substitution

$$x = a \sec(u)$$

and we'll do some of these integrals, too.