

Alternating Series

Math 1220 (Spring 2003)

An **alternating series** is a series that looks like:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

where each $a_k \geq 0$.

The test for convergence of an alternating series is particularly simple.

Alternating Series Test. If $a_1 > a_2 > a_3 > \dots$ and $\lim_{k \rightarrow \infty} a_k = 0$, then:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = L \text{ (that is, the series converges)}$$

Moreover, each partial sum $S_n = \sum_{k=1}^n (-1)^{k+1} a_k$ satisfies:

$$|S_n - L| \leq a_{n+1}$$

so if we want the limit up to an error of a_{n+1} , we need only the first n terms.

Warning: Remember! This test only applies to alternating series. There are plenty of positive series (e.g. harmonic series) that don't converge even though the sequence has 0 as a limit!

Example: The **alternating harmonic series** converges. In fact:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln(2)$$

as we shall see later.

There is one more test we ought to mention that works for any series:

The Alternating Ratio Test: Suppose we have **any** series with

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = r$$

- (i) if $r < 1$, then the alternating series converges
- (ii) if $r > 1$, then the alternating series diverges
- (iii) if $r = 1$, then this test gives no information

The idea here is the same as with the ratio test for positive series. If $r < 1$ or $r > 1$, then the series converges or diverges along with the geometric series. But if $r = 1$, then things are more delicate.

Example: The series:

$$\sum_{k=1}^{\infty} \frac{x^k}{k!}$$

converges no matter what value of x (positive or negative) we put in. We've already seen this for positive x by the positive ratio test. For negative x , we use the same test with absolute values:

$$\lim_{k \rightarrow \infty} \frac{|x^{k+1}/(k+1)!|}{|x^k/k!|} = \lim_{k \rightarrow \infty} \frac{|x|}{k+1} = 0$$

to conclude convergence by the alternating ratio test.