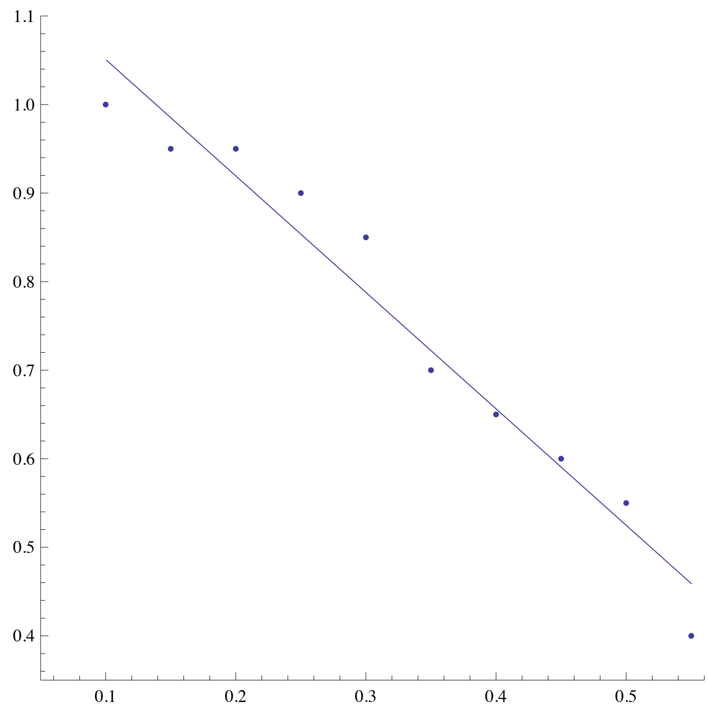
Math 5090, Assignment 9: Chapter 15, Exercises 1, 2, 8.

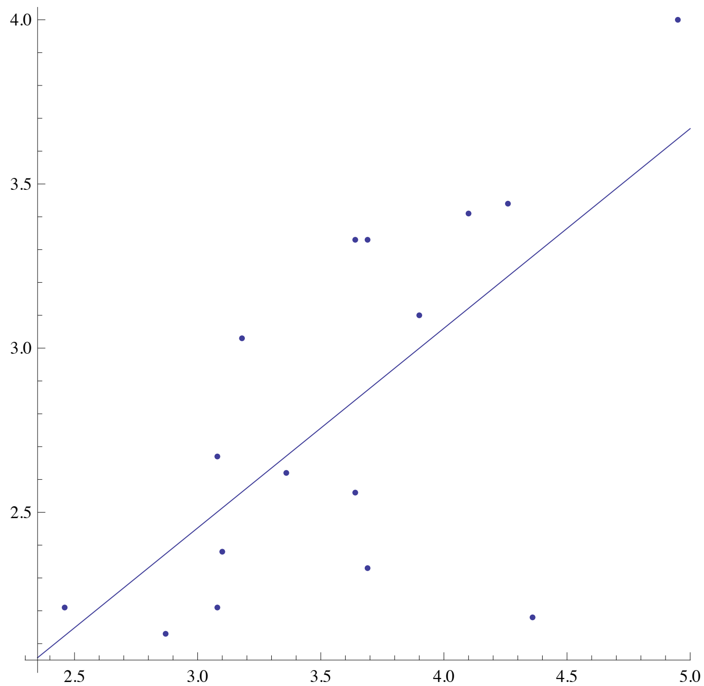
1. $n = 10$, $\overline{x} = 0.325$, $\overline{y} = 0.755$, $\sum x_i^2 = 1.2625$, $\sum x_i y_i = 2.1825$, $S_{xx} = 0.20625$, $S_{xy} = -0.27125$, $\hat{\beta}_1 = -1.31515$, and $\hat{\beta}_0 = 1.18242$. $\hat{\beta}_0 + \hat{\beta}_1(0.325) = 0.755$. $\text{SSE} = \sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = 0.0155152$, $\tilde{\sigma}^2 = \text{SSE}/8 = 0.00193939$. See figure below.



2. $n = 16$, $\overline{x} = 3.585$, $\overline{y} = 2.80813$, $\sum x_i^2 = 211.684$, $\sum x_i y_i = 164.752$, $S_{xx} = 6.0488$, $S_{xy} = 3.67785$, $\hat{\beta}_1 = 0.60803$, and $\hat{\beta}_0 = 0.628339$. $\hat{\beta}_0 + \hat{\beta}_1(4) = 3.06046$. $\text{SSE} = \sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = 2.7228$, $\tilde{\sigma}^2 = \text{SSE}/14 = 0.194486$. See figure on next page.

8. (a)

$$\begin{aligned} E[\hat{\beta}_0^2] &= \text{Var}(\hat{\beta}_0) + E[\hat{\beta}_0]^2 \\ &= \beta_0^2 + \frac{\sigma^2}{n}\,\frac{\sum x_i^2}{S_{xx}} \\ &= \beta_0^2 + \frac{\sigma^2}{n}\left(1 + \frac{n\overline{x}^2}{S_{xx}}\right). \end{aligned}$$



(b)

$$
\begin{aligned}
E[\hat{\beta}_1^2] &= \mathrm{Var}(\hat{\beta}_1) + E[\hat{\beta}_1]^2 \\
&= \beta_1^2 + \frac{\sigma^2}{S_{xx}}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
E[(Y_i - \overline{Y})^2] &= E[\sum Y_i^2 - n\overline{Y}^2] \\
&= \sum E[Y_i^2] - nE[\overline{Y}^2] \\
&= \sum (\mathrm{Var}(Y_i) + E[Y_i]^2) - n(\mathrm{Var}(\overline{Y}) + E[\overline{Y}]^2) \\
&= \sum (\sigma^2 + (\beta_0 + \beta_1 x_i)^2) - n\left(\frac{\sigma^2}{n} + (\beta_0 + \beta_1 \overline{x})^2\right) \\
&= (n-1)\sigma^2 + \sum (\beta_0 + \beta_1 x_i)^2 - n(\beta_0 + \beta_1 \overline{x})^2 \\
&= (n-1)\sigma^2 + n\beta_0^2 + 2n\beta_0\beta_1\overline{x} + \beta_1^2 \sum x_i^2 - n\beta_0^2 - 2n\beta_0\beta_1\overline{x} + n\beta_1^2\overline{x}^2 \\
&= (n-1)\sigma^2 + \beta_1^2 S_{xx}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
(n-2)E[\tilde{\sigma}^2] &= E[\sum(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2] \\
&= E[\sum(Y_i - \overline{Y} - \hat{\beta}_1(x_i - \overline{x}))^2] \\
&= E[\sum(Y_i - \overline{Y})^2] + E[\hat{\beta}_1^2]S_{xx} - 2E[\sum(Y_i - \overline{Y})\hat{\beta}_1(x_i - \overline{x})] \\
&= (n-1)\sigma^2 + \left(\beta_1^2 + \frac{\sigma^2}{S_{xx}}\right)S_{xx} - 2E[\hat{\beta}_1 S_{xy}] \\
&= (n-1)\sigma^2 + \left(\beta_1^2 + \frac{\sigma^2}{S_{xx}}\right)S_{xx} - 2E[\hat{\beta}_1^2 S_{xx}] \\
&= (n-1)\sigma^2 + 2\beta_1^2 S_{xx} + \sigma^2 - 2\left(\beta_1^2 + \frac{\sigma^2}{S_{xx}}\right)S_{xx} \\
&= (n-2)\sigma^2.
\end{aligned}
$$