Math 5090, Assignment 7: Chapter 14, Exercises 11, 13, 15, 16.

11. (a) The $B-A$ differences are -3, 10, -10, 8, -25, -2, -17, -3, -2, -2. The sample mean and variance are $\overline{x} = -4.6$ and $s^2 = 110.711$. To test $H_0 : \mu_D = 0$ vs. $H_a : \mu_D < 0$, we use the test statistic $(\overline{x} - 0)/(s/\sqrt{10}) = -1.38249$. The critical value is $-t_{0.90}(9) = -1.383$. The observed value is not less than the critical value, so we cannot reject H_0 at the $\alpha = 0.10$ level, but it is very close.

(b) We consider only the signs of the differences. 8 are negative, 2 are positive. The number of negative signs under H_0 is $\text{BIN}(10, 1/2)$, and we reject H_0 in favor of the alternative if this number is too large. The p-value is $P(\text{BIN}(10, 1/2) \geq 8) = 1 - P(\text{BIN}(10, 1/2) \leq 7) = 1 - 0.9453 = 0.0547$. This is less that 0.10, so we reject H_0 at the $\alpha = 0.10$ level.

13. (a) Note that the lambs are dieting to gain weight, not lose it. The weight differences (I $-$ II) are 14, 12, -6, 15, -2, 18, 14, 17, 14, 17, -7, -3. There are 8 positive signs (out of 12). We reject H_0 in favor of the alternative if this number is too large. The p-value is $P(\text{BIN}(12, 1/2) \geq 8) = 1 - P(\text{BIN}(12, 1/2) \leq 7) = 1 - 0.8062 = 0.1938$. We do not reject H_0 at the $\alpha = 0.10$ level.

(b) The absolute differences are 14, 12, 6, 15, 2, 18, 14, 17, 14, 17, 7, 3. The ranks are 7, 5, 3, 9, 1, 12, 7, 10.5, 7, 10.5, 4, 2, where we replace ties by average ranks. The sum of the ranks of negative differences is $3 + 1 + 4 + 2 = 10$. This is less that or equal to 21, the critical value for significance level $\alpha = 0.10$, so we reject H_0.

Now consider the large-sample test, even though $n = 12$. Under H_0, mean is $n(n + 1)/4 = 39$ and variance is $n(n + 1)(2n + 1)/24 = 162.5$. Standardize to get $(10 - 39)/\sqrt{162.5} = -2.27495$. This is smaller than -1.282, so we reject H_0.

15. (a) If we replace the observations by their ranks among the 20, we get

1–2–47: 7, 3, 13, 1, 10, 5, 2, 11, 4, 6.

10–2–47: 9, 17, 15, 14, 20, 18, 16, 8, 19, 12.

The sum of the x ranks is $W_x = 62$, so $U_x = W_x - 10(10+1)/2 = 62 - 55 = 7$. Reject H_0 if $U_x \leq 27$ or $U_x \geq 100 - 27$ (see Table 13B). We reject H_0.

Large-sample approximation: Under H_0, mean is $n^2/2 = 50$ and variance is $n^2(2n+1)/12 = 175$. Standardized statistic is $(7 - 50)/\sqrt{175} = -3.25049$. This is greater in absolute value than 1.645, so we reject H_0.

(b) Differences (second $-$ first) are 38, 324, 8, 418, 184, 326, 316, -52, 383, 237. Absolute differences are 38, 324, 8, 418, 184, 326, 316, 52, 383, 237. The sole negative term has rank 3 among the absolute values, so the Wilcoxon statistic is $T = 3$. Reject H_0 in favor of the two-sided alternative if $T \leq c$ or $T \geq 55 - c$, where c is chosen to give size $\alpha \leq 0.10$. From Table 12, we find that $c = 10$, so we reject H_0.

16. We use the Mann–Whitney test. Replacing observations by ranks in the combined sample, we get

Tester 1: 7, 9, 14, 5, 4, 2, 10, 6, 1, 3.

Tester 2: 15, 12, 13, 16, 11, 8, 18, 17, 19.

Thus, $W_x = 7 + 9 + 14 + 5 + 4 + 2 + 10 + 6 + 1 + 3 = 61$, so $U_x = W_x - (10)(10 + 1)/2 = 61 - 55 = 6$. The critical region for a two-sided test is, by Table 12, $U_x \leq 24$ or $U_x \geq 90 - 24$. We reject H_0.