Math 5090, Assignment 4, Chapter 12, Exercises 27, 30, 37, 39.

27. (a) $f(\boldsymbol{x};\theta) = \theta^{-n}\exp\{-\sum x_i/\theta\}$. $\log f(\boldsymbol{x};\theta) = -n\log\theta - \sum x_i/\theta$ has derivative $-n/\theta + \sum x_i/\theta^2$. Set equal to 0 and solve to get the MLE, $\hat\theta = \overline{x}$. Under H_0, $\hat\theta_0 = \theta_0$. So the GLR statistic is

$$\lambda(\boldsymbol{x}) = \frac{f(\boldsymbol{x};\hat\theta_0)}{f(\boldsymbol{x};\hat\theta)} = \frac{\theta_0^{-n}\exp\{-\sum x_i/\theta_0\}}{\overline{x}^{-n}\exp\{-\sum x_i/\overline{x}\}} = \mathrm{const}_n\,\overline{x}^n\exp\{-(n/\theta_0)\overline{x}\}.$$

With $g(x) = x^n e^{-(n/\theta_0)x}$, choose $c_1 < c_2$ such that $g(c_1) = g(c_2)$ and $\{\overline{x} \le c_1$ or $\overline{x} \ge c_2\}$ has size α. This is the GLR test of size α. It is difficult to determine the constants c_1 and c_2 precisely, so an equal-tailed test is usually chosen.

In the large sample case, we use $-2\log\lambda(\boldsymbol{x})$ as our test statistic, and an approximate critical region for the size α test is $\{-2\log\lambda(\boldsymbol{x}) \ge \chi^2_{1-\alpha}(1)\}$.

(b) We need to find the MLE when θ is restricted to $[\theta_0,\infty)$. It is $\hat\theta = \overline{x}$ if $\overline{x} \ge \theta_0$ and θ_0 if $\overline{x} < \theta_0$. In other words, it is $\max(\overline{x},\theta_0)$. Now the GLR statistic is

$$\begin{aligned}\lambda(\boldsymbol{x}) &= \frac{f(\boldsymbol{x};\hat\theta_0)}{f(\boldsymbol{x};\hat\theta)} = \frac{\theta_0^{-n}\exp\{-\sum x_i/\theta_0\}}{\max(\overline{x},\theta_0)^{-n}\exp\{-\sum x_i/\max(\overline{x},\theta_0)\}}\\ &= \begin{cases}\mathrm{const}_n\,\overline{x}^n\exp\{-(n/\theta_0)\overline{x}\} & \text{if } \overline{x}\ge\theta_0\\ 1 & \text{if } \overline{x}<\theta_0\end{cases}.\end{aligned}$$

Then $\lambda(\boldsymbol{x}) \le k$ is equivalent to $\overline{x} \ge c$, and c can be chosen to give size α. Specifically, $\overline{x}$ is $\mathrm{GAM}(\theta_0/n, n)$ under H_0 so $(2n/\theta_0)\overline{x}$ is $\mathrm{GAM}(2,n)$ or $\chi^2(2n)$. We reject H_0 if $(2n/\theta_0)\overline{x} \ge \chi^2_{1-\alpha}(2n)$.

30. (a) Assume means are known. The likelihood function is $L(\sigma_1^2,\sigma_2^2) = (2\pi\sigma_1^2)^{-n_1/2}\exp\{-\sum(x_i-\mu_1)^2/(2\sigma_1^2)\}(2\pi\sigma_2^2)^{-n_2/2}\exp\{-\sum(y_i-\mu_2)^2/(2\sigma_2^2)\}$, so we can find the unrestricted MLEs to be $\hat\sigma_1^2 = (1/n_1)\sum(x_i-\mu_1)^2$ and $\hat\sigma_2^2 = (1/n_2)\sum(y_i-\mu_2)^2$.

Next we must find the MLEs under H_0. Under H_0 the likelihood function is $L_0(\sigma^2) = (2\pi\sigma^2)^{-(n_1+n_2)/2}\exp\{-\sum(x_i-\mu_1)^2/(2\sigma^2)\}\exp\{-\sum(y_i-\mu_2)^2/(2\sigma^2)\}$. We get $\hat\sigma_0^2 = [\sum(x_i-\mu_1)^2 + \sum(y_i-\mu_2)^2]/(n_1+n_2)$, or

$$\hat\sigma_0^2 = \frac{n_1}{n_1+n_2}\hat\sigma_1^2 + \frac{n_2}{n_1+n_2}\hat\sigma_2^2.$$

So our GLR statistic $\lambda(\boldsymbol{x},\boldsymbol{y})$ is

$$\begin{aligned}&\frac{L_0(\hat\sigma_0^2)}{L(\hat\sigma_1^2,\hat\sigma_2^2)}\\ &= \frac{(2\pi\hat\sigma_0^2)^{-(n_1+n_2)/2}\exp\{-\sum(x_i-\mu_1)^2/(2\hat\sigma_0^2)\}\exp\{-\sum(y_i-\mu_2)^2/(2\hat\sigma_0^2)\}}{(2\pi\hat\sigma_1^2)^{-n_1/2}\exp\{-\sum(x_i-\mu_1)^2/(2\hat\sigma_1^2)\}(2\pi\hat\sigma_2^2)^{-n_2/2}\exp\{-\sum(y_i-\mu_2)^2/(2\hat\sigma_2^2)\}}\\ &= \frac{(\hat\sigma_0^2)^{-(n_1+n_2)/2}}{(\hat\sigma_1^2)^{-n_1/2}(\hat\sigma_2^2)^{-n_2/2}}.\end{aligned}$$

This is a function g of $\hat\sigma_1^2/\hat\sigma_2^2$, and $g(x) \le k$ is equivalent to $x \le c_1$ or $x \ge c_2$. Again it is difficult to determine the exact c_1 and c_2, so an equal-tailed test is usually used. Under H_0, $\hat\sigma_1^2/\hat\sigma_2^2$ has an $F(n_1, n_2)$ distribution, so we reject H_0 if this statistic is $\le F_{\alpha/2}(n_1, n_2)$ or $\ge F_{1-\alpha/2}(n_1, n_2)$.

(b) Here the means are unknown. The unrestricted MLEs are $\hat\mu_1 = \overline{x}$, $\hat\mu_2 = \overline{y}$, $\hat\sigma_1^2 = n_1^{-1}\sum(x_i - \overline{x})^2$, and $\hat\sigma_2^2 = n_2^{-1}\sum(y_i - \overline{y})^2$. Under H_0, the MLEs are $\hat\mu_1 = \overline{x}$, $\hat\mu_2 = \overline{y}$, and $\hat\sigma_0^2 = (n_1+n_2)^{-1}[\sum(x_i - \overline{x})^2 + \sum(y_i - \overline{y})^2]$. So our GLR statistic $\lambda(\boldsymbol{x}, \boldsymbol{y})$ is

$$
\begin{aligned}
&\frac{L_0(\hat\mu_1, \hat\mu_2, \hat\sigma_0^2)}{L(\hat\mu_1, \hat\mu_2, \hat\sigma_1^2, \hat\sigma_2^2)} \\
&= \frac{(2\pi\hat\sigma_0^2)^{-(n_1+n_2)/2}\exp\{-\sum(x_i-\overline{x})^2/(2\hat\sigma_0^2)\}\exp\{-\sum(y_i-\overline{y})^2/(2\hat\sigma_0^2)\}}{(2\pi\hat\sigma_1^2)^{-n_1/2}\exp\{-\sum(x_i-\overline{x})^2/(2\hat\sigma_1^2)\}(2\pi\hat\sigma_2^2)^{-n_2/2}\exp\{-\sum(y_i-\overline{y})^2/(2\hat\sigma_2^2)\}} \\
&= \frac{(\hat\sigma_0^2)^{-(n_1+n_2)/2}}{(\hat\sigma_1^2)^{-n_1/2}(\hat\sigma_2^2)^{-n_2/2}}.
\end{aligned}
$$

The situation is similar to part (a). Again we use an equal-tailed test. Under H_0, $\hat\sigma_1^2/\hat\sigma_2^2$ has an $F(n_1-1, n_2-1)$ distribution, so we reject H_0 if this statistic is $\le F_{\alpha/2}(n_1-1, n_2-1)$ or $\ge F_{1-\alpha/2}(n_1-1, n_2-1)$.

(c) The unrestricted MLEs are as in part (b). Under H_0, the MLEs are those for a single sample of size n_1+n_2, so $\hat\mu = (n_1\overline{x} + n_2\overline{y})/(n_1+n_2)$ and $\hat\sigma^2 = (n_1+n_2)^{-1}[\sum(x_i - \hat\mu)^2 + \sum(y_i - \hat\mu)^2]$. So our GLR statistic $\lambda(\boldsymbol{x}, \boldsymbol{y})$ is

$$
\begin{aligned}
&\frac{L_0(\hat\mu, \hat\sigma^2)}{L(\hat\mu_1, \hat\mu_2, \hat\sigma_1^2, \hat\sigma_2^2)} \\
&= \frac{(2\pi\hat\sigma^2)^{-(n_1+n_2)/2}\exp\{-\sum(x_i-\hat\mu)^2/(2\hat\sigma^2)\}\exp\{-\sum(y_i-\hat\mu)^2/(2\hat\sigma^2)\}}{(2\pi\hat\sigma_1^2)^{-n_1/2}\exp\{-\sum(x_i-\overline{x})^2/(2\hat\sigma_1^2)\}(2\pi\hat\sigma_2^2)^{-n_2/2}\exp\{-\sum(y_i-\overline{y})^2/(2\hat\sigma_2^2)\}} \\
&= \frac{(\hat\sigma^2)^{-(n_1+n_2)/2}}{(\hat\sigma_1^2)^{-n_1/2}(\hat\sigma_2^2)^{-n_2/2}}.
\end{aligned}
$$

We reject H_0 for small values of this statistic. Since its null distribution is unclear, we use the large-sample approximation, reject H_0 if $-2\log\lambda(\boldsymbol{x}, \boldsymbol{y}) \ge \chi^2_{1-\alpha}(4-2)$.

37. (a) $k_0^* = \alpha/(1-\beta) = 0.10/0.95 = 0.1053$ and $k_1^* = (1-\alpha)/\beta = 0.90/0.05 = 18$.

(b) The nth likelihood ratio is

$$
L_n(\boldsymbol{x}) = \frac{(2\pi)^{-n/2}\exp\{-(1/2)\sum_1^n x_i^2\}}{(18\pi)^{-n/2}\exp\{-(1/18)\sum_1^n x_i^2\}} = 3^n \exp\{-(4/9)\sum_1^n x_i^2\}
$$

If, for some $n \ge 1$, $k_0^* < L_m(\boldsymbol{x}) < k_1^*$ for $m = 1, 2, \ldots, n-1$ and $L_n(\boldsymbol{x}) \le k_0^*$, then we reject H_0. If, for some $n \ge 1$, $k_0^* < L_m(\boldsymbol{x}) < k_1^*$ for $m = 1, 2, \ldots, n-1$ and $L_n(\boldsymbol{x}) \ge k_1^*$, then we accept H_0.

(c) Take logs in part (b). $\log L_n(\boldsymbol{x}) = n\log 3 - (4/9)\sum_1^n x_i^2$. If, for some $n \ge 1$, $\log k_0^* < m\log 3 - (4/9)\sum_1^m x_i^2 < \log k_1^*$ for $m = 1, 2, \ldots, n-1$ and

$n\log 3-(4/9)\sum_1^n x_i^2 \le k_0^*$, then we reject H_0. If, for some $n \ge 1$, $\log k_0^* < m\log 3-(4/9)\sum_1^m x_i^2 < \log k_1^*$ for $m=1,2,\ldots,n-1$ and $n\log 3-(4/9)\sum_1^n x_i^2 \ge k_1^*$, then we accept H_0.

(d) $E_{\sigma=1}[N] = \{\alpha\log[\alpha/(1-\beta)]+(1-\alpha)\log[(1-\alpha)/\beta]\}/E_{\sigma=1}[\log 3-(4/9)x_1^2] = (0.10\log[0.10/0.95]+0.90\log[0.90/0.05])/(\log 3-(4/9)) = 3.63$ and $E_{\sigma=3}[N] = \{(1-\beta)\log[\alpha/(1-\beta)]+\beta\log[(1-\alpha)/\beta]\}/E_{\sigma=3}[\log 3-(4/9)x_1^2] = (0.95\log[0.10/0.95]+0.05\log[0.90/0.05])/(\log 3-(4/9)3^2) = 0.69$.

(e) $\log k_0^* = -2.25$ and $\log k_1^* = 2.89$. If $(x_1,x_2,\ldots,x_{10}) = (-2.20, 0.50, 2.55, -1.85, -0.45, -1.15, -0.58, 5.65, 0.49, -1.16)$, then $(s_1,\ldots,s_{10}) = (-1.05, -0.0650, -1.86, -2.28, -1.27, -0.759, 0.190, -12.9, -11.9, -11.4)$. This exits from the interval at the fourth step, so the test terminates in four steps with rejection of H_0.

39. (a) The nth likelihood ratio is

$$L_n(\boldsymbol{x}) = \frac{\theta_0^{-n}\exp\{-\sum_1^n x_i/\theta_0\}}{\theta_1^{-n}\exp\{-\sum_1^n x_i/\theta_1\}} = \frac{2^{-n}\exp\{-\sum_1^n x_i/2\}}{4^{-n}\exp\{-\sum_1^n x_i/4\}} = 2^n\exp\{-\sum_1^n x_i/4\}$$

so $\log L_n(\boldsymbol{x}) = n\log 2-\sum_1^n x_i/4$. In particular $z_1 = \log 2 - x_1/4$. With $(x_1,x_2,\ldots,x_{20}) = (1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0)$. As in Exercise 37, $\log k_0^* = -2.25$ and $\log k_1^* = 2.89$. At the 14th step, S_n exits from the interval for the first time with $S_{14} = 3.2$. Thus, we fail to reject H_0.

(b) $E_{\theta=2}[N] = \{\alpha\log[\alpha/(1-\beta)]+(1-\alpha)\log[(1-\alpha)/\beta]\}/E_{\theta=2}[\log 2-x_1/4] = (0.10\log[0.10/0.95]+0.90\log[0.90/0.05])/(\log 2-2/4) = 12.3$

(c) $E_{\theta=4}[N] = \{(1-\beta)\log[\alpha/(1-\beta)]+\beta\log[(1-\alpha)/\beta]\}/E_{\sigma=3}[\log 2-x_1/4] = (0.95\log[0.10/0.95]+0.05\log[0.90/0.05])/(\log 2-4/4) = 6.5$.