Math 5090, Assignment 3, Chapter 12, Exercises 17, 18, 22, 24.

17. (a) For $\sigma_2 > \sigma_1 > 0$ the likelihood ratio is

$$\frac{(2\pi\sigma_2^2)^{-n/2}\exp\{-(1/\sigma_2^2)\sum_1^n x_i^2\}}{(2\pi\sigma_1^2)^{-n/2}\exp\{-(1/\sigma_1^2)\sum_1^n x_i^2\}} = \left(\frac{\sigma_2^2}{\sigma_1^2}\right)^{-n/2}\exp\{-(1/\sigma_2^2 - 1/\sigma_1^2)\sum_1^n x_i^2\},$$

which is increasing in $\sum_1^n x_i^2$. By the MLR theorem, a UMP test of $H_0 : \sigma = \sigma_0$ vs. $H_a : \sigma > \sigma_0$ has the form: Reject H_0 if $\sum_1^n x_i^2 \geq k$, where k is chosen to assure size α. Under H_0, $\sum_1^n x_i^2/\sigma_0^2$ is $\chi^2(n)$, so we reject H_0 if $\sum_1^n x_i^2 \geq \sigma_0^2\chi_{1-\alpha}^2(n)$.

(b) $\pi(\sigma) = P_\sigma(\sum_1^n x_i^2 \geq \sigma_0^2\chi_{1-\alpha}^2(n)) = P(\chi^2(n) \geq (\sigma_0^2/\sigma^2)\chi_{1-\alpha}^2(n))$. Note that this is α when $\sigma = \sigma_0$.

(c) $\pi(2) = P(\chi^2(20) \geq (1/4)\chi_{0.995}^2(20)) = P(\chi^2(20) \geq (1/4)40.00) = P(\chi^2(20) \geq 10) = 1 - 0.032 = 0.968$.

18. Let $\theta_1 < \theta_0$. First consider $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_1$. Then the NP statistic is

$$\lambda(\boldsymbol{x};\theta_0,\theta_1) = \frac{(1/\theta_0)^n 1_{(0,\theta_0)}(x_{n:n})}{(1/\theta_1)^n 1_{(0,\theta_1)}(x_{n:n})}$$

We reject H_0 if this is $\leq k$, or equivalently if $1/\lambda(\boldsymbol{x};\theta_0,\theta_1) \geq 1/k$. Notice that $1/\lambda(\boldsymbol{x};\theta_0,\theta_1)$ is nonincreasing in $x_{n:n}$, so a nearly equivalent rejection criterion is $x_{n:n} \leq k_1$. Under H_0, $x_{n:n}/\theta_0$ has CDF $P_{\theta_0}(x_{n:n}/\theta_0 \leq x) = x^n$, which is α if $x = \alpha^{1/n}$. So our critical region is $x_{n:n} \leq \theta_0\alpha^{1/n}$.

Now this critical region does not depend on θ_1 (only on the fact that $\theta_1 < \theta_0$), so the test is UMP for $H_0 : \theta = \theta_0$ vs. $H_a : \theta < \theta_0$.

To extend to the composite null hypothesis, we need only show that the power of the test for $\theta \geq \theta_0$ is maximized at θ_0. For this, $\pi(\theta) = P_\theta(x_{n:n} \leq \theta_0\alpha^{1/n}) = P_\theta(x_{n:n}/\theta \leq (\theta_0/\theta)\alpha^{1/n}) = [(\theta_0/\theta)\alpha^{1/n}]^n = (\theta_0/\theta)^n\alpha$, from which the desired result follows.

22. (a) For $0 < \theta_1 < \theta_2$, the likelihood ratio is

$$\frac{(\theta_2^\kappa\Gamma(\kappa))^{-n}(x_1\cdots x_n)^{\kappa-1}\exp\{-\sum x_i/\theta_2\}}{(\theta_1^\kappa\Gamma(\kappa))^{-n}(x_1\cdots x_n)^{\kappa-1}\exp\{-\sum x_i/\theta_1\}} = \left(\frac{\theta_1}{\theta_2}\right)^{\kappa n}\exp\left\{-(1/\theta_2 - 1/\theta_1)\sum_1^n x_i\right\},$$

which is increasing in $\sum_1^n x_i$. This has the MLR property, so a UMP test is of the form: Reject H_0 if $\sum_1^n x_i \geq k$. Now, if $\theta = \theta_0$, then $\sum_1^n x_i$ has distribution $\mathrm{GAM}(\theta_0, n\kappa)$, so $(2/\theta_0)\sum_1^n x_i$ is $\mathrm{GAM}(2, 2n\kappa/2)$ or $\chi^2(2n\kappa)$. We reject H_0 if $(2/\theta_0)\sum_1^n x_i \geq \chi_{1-\alpha}^2(2n\kappa)$.

(b) $\pi(\theta) = P_\theta((2/\theta_0)\sum_1^n x_i \geq \chi_{1-\alpha}^2(2n\kappa)) = P_\theta((2/\theta)\sum_1^n x_i \geq (\theta_0/\theta)\chi_{1-\alpha}^2(2n\kappa)) = P(\chi^2(2n\kappa) \geq (\theta_0/\theta)\chi_{1-\alpha}^2(2n\kappa))$.

(c) $\pi(2) = P(\chi^2(16) \geq (1/2)\chi_{0.99}^2(16)) = P(\chi^2(16) \geq (1/2)32.00) = P(\chi^2(16) \geq 16) = 1 - 0.547 = 0.453$.

(d) If $0 < \kappa_1 < \kappa_2$, the likelihood ratio is

$$\frac{(\theta^{\kappa_2}\Gamma(\kappa_2))^{-n}(x_1\cdots x_n)^{\kappa_2-1}\exp\{-\sum x_i/\theta\}}{(\theta^{\kappa_1}\Gamma(\kappa_1))^{-n}(x_1\cdots x_n)^{\kappa_1-1}\exp\{-\sum x_i/\theta\}} = \frac{(\theta^{\kappa_2}\Gamma(\kappa_2))^{-n}}{(\theta^{\kappa_1}\Gamma(\kappa_1))^{-n}}(x_1\cdots x_n)^{\kappa_2-\kappa_1},$$

which is an increasing function of $x_1\cdots x_n$. This has the MLR property, so a UMP test is of the form: Reject H_0 if $x_1\cdots x_n \geq k$. Unfortunately, we cannot go any further with this because we do not know the distribution of the product of n independent GAM(θ, κ_0) random variables.

24. For $0 < \theta_1 < \theta_2$, the likelihood ratio is

$$\frac{2^n\theta_2^{-2n}x_1\cdots x_n\exp\{-\sum(x_i/\theta_2)^2\}}{2^n\theta_1^{-2n}x_1\cdots x_n\exp\{-\sum(x_i/\theta_1)^2\}} = \left(\frac{\theta_1}{\theta_2}\right)^{2n}\exp\left\{-(1/\theta_2^2 - 1/\theta_1^2)\sum_1^n x_i^2\right\},$$

which is an increasing function of $\sum_1^n x_i^2$. This has the MLR property, so a UMP test is of the form: Reject H_0 if $\sum_1^n x_i^2 \leq k$. If X is WEI$(\theta, 2)$, then X^2 has density EXP(θ^2). Hence, if $\theta = \theta_0$, $\sum_1^n x_i^2$ is GAM(θ_0^2, n) and $(2/\theta_0^2)\sum_1^n x_i^2$ is GAM$(2, 2n/2) = \chi^2(2n)$. We reject H_0 if $(2/\theta_0^2)\sum_1^n x_i^2 \leq \chi^2_\alpha(2n)$.