Math 5090, Assignment 2, Chapter 12, Exercises 6, 9, 11, 16.

6. (a) $P_{\eta_0}(X_1 \geq c) = e^{-(c-\eta_0)}$ if $c \geq \eta_0$, so $P_{\eta_0}(X_{1:n} \geq c) = e^{-n(c-\eta_0)}$ if $c \geq \eta_0$. For this to be α we need $c = -n^{-1}\log\alpha + \eta_0$.

(b) $\pi(\eta) = P_\eta(X_{1:n} \geq c) = e^{-n(c-\eta)} = e^{-n(-n^{-1}\log\alpha+\eta_0-\eta)}$ as long as the quantity in parentheses is nonnegative, so $\pi(\eta) = \min(\alpha e^{n(\eta-\eta_0)}, 1)$.

(c) We want $\pi(\eta_1) = 1-\beta$ so $\alpha e^{n(\eta_1-\eta_0)} = 1-\beta$ or $n(\eta_1-\eta_0) = \log(1-\beta) - \log\alpha$, hence $n = (\log(1-\beta)-\log\alpha)/(\eta_1-\eta_0)$.

9. (a) $S_p^2 = [(n_1-1)S_1^2 + (n_2-1)S_2^2]/(n_1+n_2-2) = (8\cdot 36 + 8\cdot 45)/(16) = 40.5$. The text statistic is $T_0 = (\overline{Y}-\overline{X})/(S_p\sqrt{1/n_1+1/n_2}) = (10-16)/\sqrt{40.5(1/9+1/9)} = -2$. But $t_{0.95}(16) = 1.746$, so since $|-2| > 1.746$ we reject H_0.

(b) $T = (\overline{Y}-\overline{X})/\sqrt{S_1^2/n1 + S_2^2/n_2} = (10-16)/\sqrt{36/9+45/9} = -2$ and degrees of freedom are estimated as

$$\frac{(s_1^2/n_1+s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1-1)+(s_2^2/n_2)^2/(n_2-1)} = \frac{(36/9+45/9)^2}{(36/9)^2/8+(45/9)^2/8} = \frac{81}{41/8} = 15.8,$$

so we interpolate to get about 1.75, and we again reject H_0.

(c) $T = (\overline{Y}-\overline{X})/(S_D/\sqrt{n}) = (10-16)/\sqrt{81/9} = -2$ and $t_{0.95}(8) = 1.860$. Since $|-2| > 1.860$ we reject H_0.

(d) $F_0 = S_1^2/S_2^2 = 36/45 = 0.8$. $F_{0.95}(8,8) = 3.44$. Since $0.8 > 1/3.44 = 0.291$, we fail to reject H_0.

(e) $P_{\sigma_2^2/\sigma_1^2=1.33}(S_1^2/S_2^2 \leq 1/3.44) = P((S_1^2/\sigma_1^2)/(S_2^2/\sigma_2^2) \leq 1.33/3.44) = P(F(8,8) \leq 0.3866) = P(F(8,8) \geq 2.586) = 0.10$ from Table 7.

11. (a) $\lambda(x,1,2) = 1/(2x)$ is the NP statistic. Under H_0, X has a UNIF$(0,1)$ distribution. $P_1(1/(2X) \leq k) = P_1(X \geq c) = 0.05$ if $c = 0.95$. Reject H_0 if $X \geq 0.95$.

(b) $\pi(2) = P_2(X \geq 0.95) = \int_{0.95}^1 2x\,dx = 1 - 0.95^2 = 0.0975$.

(c) $\lambda(\boldsymbol{x},1,2) = 1/(2^n x_1 x_2 \cdots x_n)$, so the critical region has the form $\{\boldsymbol{x} : x_1x_2\cdots x_n \geq c\}$. We need to determine c to have size $\alpha = 0.05$. Under H_0, $X_1, X_2, \ldots, X_n$ are independent UNIF$(0,1)$, hence $-2\log X_1, \ldots, -2\log X_n$ are independent EXP(2) and the critical region is $\{\boldsymbol{x} : -2\sum_1^n \log x_i \leq c_1\}$. The sum of n independent EXP(2) random variables is GAM$(2,n) = \chi^2(2n)$. So $c_1 = \chi^2_{0.05}(2n)$ gives the desired size.

16. We can apply the NP Lemma with $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_1$ with $\theta_1 > \theta_0$.

$$\lambda(\boldsymbol{x},\theta_0,\theta_1) = \frac{(3^n x_1^2\cdots x_n^2/\theta_0^n)e^{-\sum_1^n x_i^3/\theta_0}}{(3^n x_1^2\cdots x_n^2/\theta_1^n)e^{-\sum_1^n x_i^3/\theta_1}} = \frac{\theta_1^n}{\theta_0^n}e^{-(1/\theta_0-1/\theta_1)\sum_1^n x_i^3}.$$

This is $\leq k$ if and only if $\sum_1^n x_i^3 \geq k_1$, since $1/\theta_0 - 1/\theta_1 > 0$. What is the distribution of X^3 if X has density $f(x;\theta)$? The transformation formula tells us it is EXP(θ). So our test statistic, under H_0, can be taken to be $(2/\theta_0)\sum_1^n X_i^3$, which is GAM$(2,n) = \chi^2(2n)$. We reject H_0 if $(2/\theta_0)\sum_1^n X_i^3 \geq \chi^2_{1-\alpha}(2n)$. By the NP Lemma, this is UMP.