Math 5090, Assignment 11: Chapter 15, Exercises 17, 28, 33.

17. Here we can use the results of Theorem 15.4.4, generalized slightly so that the first column of $\boldsymbol{X}$ is not necessarily all 1s.

(a) Let $\boldsymbol{X}$ be the column vector consisting of $x_1, x_2, \ldots, x_n$. Then $\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = \sum_{i=1}^n x_i y_i / \sum_{i=1}^n x_i^2$ and $\hat{\sigma}^2 = (1/n)\|\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\|^2 = (1/n)\sum_{i=1}^n (y_i - \hat{\beta}x_i)^2$.

(b) $\hat{\beta} \sim N(\beta, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}) = N(\beta, \sigma^2/\sum_{i=1}^n x_i^2)$.

(c) This is a conclusion of the theorem.

(d) $\tilde{\sigma}^2 = n\hat{\sigma}^2/(n-1)$ is unbiased and $(n-1)\tilde{\sigma}^2 \sim \chi^2(n-1)$.

(e) $(\hat{\beta} - \beta)/\sqrt{\tilde{\sigma}^2/\sum_{i=1}^n x_i^2} \sim t(n-1)$.

(f) $\hat{\beta} \pm t_{1-\alpha/2}(n-1)\sqrt{\tilde{\sigma}^2/\sum_{i=1}^n x_i^2}$.

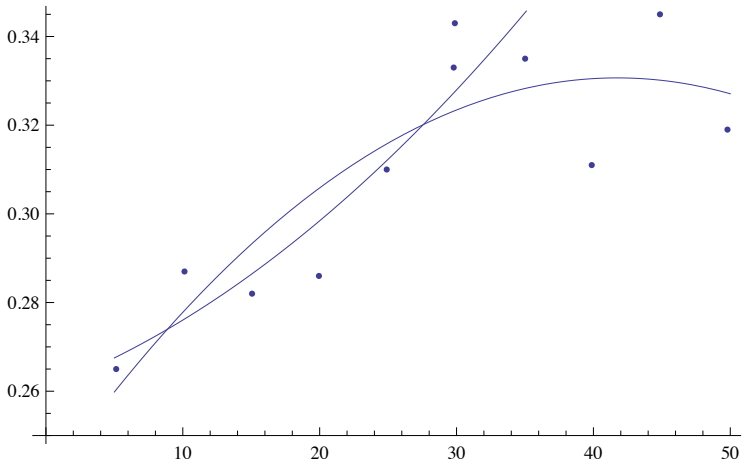
(g) $((n-1)\tilde{\sigma}^2/\chi^2_{1-\alpha/2}(n-1), (n-1)\tilde{\sigma}^2/\chi^2_{\alpha/2}(n-1))$.

28. (a) Let

$$\boldsymbol{X} = \begin{pmatrix} 1 & 5.133 & 5.133^2 \\ 1 & 10.124 & 10.124^2 \\ 1 & 15.060 & 15.060^2 \\ 1 & 19.946 & 19.946^2 \\ 1 & 24.899 & 24.899^2 \\ 1 & 29.792 & 29.792^2 \\ 1 & 29.877 & 29.877^2 \\ 1 & 35.011 & 35.011^2 \end{pmatrix}$$

Then $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = (0.260759, 0.00117258, 0.0000355693)'$ and $\tilde{\sigma}^2 = \text{SSE}/(8-3) = 0.000135606$.

(b) $\hat{y} = 0.260759 + 0.00117258x + 0.0000355693x^2$. Cf. $\hat{y} = 0.239231 + 0.00438109x - 0.0000524809x^2$ for all 11 points.



No, it does not make sense to predict past the range of the data.

(c) $\hat{\beta}_0 \pm t_{0.975}(5)\sqrt{\tilde{\sigma}^2 a_{11}} = 0.260759 \pm 0.0472001$, or $(0.213559, 0.307959)$.
Here a_{11} is the $(1,1)$ entry of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$.

(d) $\hat{\beta}_1 \pm t_{0.975}(5)\sqrt{\tilde{\sigma}^2 a_{22}} = 0.00117258 \pm 0.00537962$, or $(-0.00420704, 0.0065522)$.

(e) $\hat{\beta}_2 \pm t_{0.975}(5)\sqrt{\tilde{\sigma}^2 a_{33}} = 0.0000355693 \pm 0.000131491$, or $(-0.0000959217, 0.00016706)$.

(f) $((8-3)\tilde{\sigma}^2/\chi^2_{0.975}(8-3), (8-3)\tilde{\sigma}^2/\chi^2_{0.975}(8-3)) = (0.0000528, 0.000817)$.

33. From calculations done for Problem 2 of Chapter 15, we have $S_{xx} = 6.0488$, $S_{xy} = 3.67785$, and $S_{yy} = 4.95904$, hence $r = S_{xy}/\sqrt{S_{xx}S_{yy}} = 0.671522$. Hence $t = \sqrt{n-2}\, r/\sqrt{1-r^2} = 3.3909$. This is greater in absolute value than $t_{0.95}(16-2) = 1.761$, so we reject H_0.