Math 5090, Assignment 1, Chapter 12, Exercises 1, 2, 3, 5.

1. (a) The critical region $A = \{\overline{x} \leq a\}$ will have size $\alpha = 0.05$ if $P_{\mu=20}(\overline{X} \leq a) = P_{\mu=20}((\overline{X} - 20)/(1/\sqrt{16}) \leq (a - 20)/(1/\sqrt{16})) = 0.05$, which is equivalent to $(a - 20)/(1/\sqrt{16}) = -1.645$, so $a = 20 - 1.645/4 = 19.589$. The critical region $B = \{\overline{x} \geq b\}$ will have size $\alpha = 0.05$ if $P_{\mu=20}(\overline{X} \geq b) = P_{\mu=20}((\overline{X} - 20)/(1/\sqrt{16}) \geq (b - 20)/(1/\sqrt{16})) = 0.05$, or if $(b - 20)/(1/\sqrt{16}) = 1.645$, so $b = 20 + 1.645/4 = 20.411$.

(b) A: $\beta = P_{\mu=21}(\overline{X} > a) = P_{\mu=21}((\overline{X}-21)/(1/\sqrt{16}) > (a-21)/(1/\sqrt{16})) = 1 - \Phi((a - 21)/(1/\sqrt{16})) = 1 - \Phi(4(19.589 - 21)) = 1 - \Phi(-5.644) \approx 1$. B: $\beta = P_{\mu=21}(\overline{X} < b) = P_{\mu=21}((\overline{X} - 21)/(1/\sqrt{16}) < (b - 21)/(1/\sqrt{16})) = \Phi((b - 21)/(1/\sqrt{16})) = \Phi(4(20.411 - 21)) = \Phi(-2.356) = 1 - \Phi(2.356) \approx 0.0092$. For the alternative $\mu = 21$, A is unreasonable, as the large type II error suggests.

(c) A: $\beta = P_{\mu=19}(\overline{X} > a) = P_{\mu=19}((\overline{X}-19)/(1/\sqrt{16}) > (a-19)/(1/\sqrt{16})) = 1 - \Phi((a - 19)/(1/\sqrt{16})) = 1 - \Phi(4(19.589 - 19)) = 1 - \Phi(2.356) \approx 0.0092$. B: $\beta = P_{\mu=19}(\overline{X} < b) = P_{\mu=19}((\overline{X} - 19)/(1/\sqrt{16}) < (b - 19)/(1/\sqrt{16})) = \Phi((b - 19)/(1/\sqrt{16})) = \Phi(4(20.411 - 19)) = \Phi(5.644) \approx 1$. For the alternative $\mu = 19$, B is unreasonable, as the large type II error suggests.

(d) It is $P_{\mu=20}(\overline{X} \leq a \text{ or } \overline{X} \geq b) = P_{\mu=20}(\overline{X} \leq a) + P_{\mu=20}(\overline{X} \geq b) = \alpha + \alpha = 2\alpha = 0.10$.

(e) It is the same for $\mu = 19$ or $\mu = 21$, by symmetry. $\beta = 1 - P_{\mu=21}(\overline{X} \leq a \text{ or } \overline{X} \geq b) = 1 - P_{\mu=21}(\overline{X} \leq a) + P_{\mu=21}(\overline{X} \geq b) = 1 - (1 - \beta_A) - (1 - \beta_B) \approx 1 - (1 - 1) - (1 - 0.0092) \approx 0.0092$.

2. (a) Notice that θ is a discrete parameter here, which is unusual. The probability of a type I error is the probability of rejecting H_0 when it is true, so it is the probability of getting 2 whites or 2 blacks in the sample of two, when there are two of each in the box. This is $(\theta/4)^2 + (1 - \theta/4)^2$ when $\theta = 2$, which equals 1/2.

(b) The probability of failing to reject H_0 is $\beta = 2(\theta/4)(1 - \theta/4)$. If $\theta = 0$ or 4, $\beta = 0$. If $\theta = 1$ or 3, $\beta = 3/8$.

(c) The probability of rejecting H_0 is no longer $(\theta/4)^2 + (1-\theta/4)^2$ but rather $\theta(\theta - 1)/(4 \cdot 3) + (4 - \theta)(3 - \theta)/(4 \cdot 3)$. When $\theta = 2$, this equals $1/6 + 1/6 = 1/3$. The probability of failing to reject H_0 is now $\beta = 2\theta(4 - \theta)/(4 \cdot 3)$. If $\theta = 0$ or 4, $\beta = 0$. If $\theta = 1$ or 3, $\beta = 1/2$.

3. (a) Test statistic is $T_0 = (\overline{X} - 12)/(\sigma/\sqrt{20}) = (11 - 12)/(2/\sqrt{20}) = -\sqrt{5} = -2.236$. We reject H_0 if this is less than $-z_{0.99} = -2.326$ (p. 603). It is not, so we do not reject H_0 at the 0.01 level.

(b) $\beta = P_{\mu=10.5}(T_0 > -2.326) = P_{\mu=10.5}((\overline{X} - 10.5)/(\sigma/\sqrt{20}) > -2.326 + 1.5/(\sigma/\sqrt{20})) = 1 - \Phi(-2.326 + 1.5\sqrt{5}) = 1 - \Phi(1.0281) = 0.152$.

(c) power $= P_{\mu=10.5}(T_0 \leq -2.326) = P_{\mu=10.5}((\overline{X}-10.5)/(\sigma/\sqrt{n}) \leq -2.326 + 1.5/(\sigma/\sqrt{n})) = \Phi(-2.326 + 0.75\sqrt{n}) \geq 0.90$ if $-2.326 + 0.75\sqrt{n} \geq 1.282$ or $\sqrt{n} \geq 4.81$ or $n \geq 23.1$.

(d) Test statistic is $T_0 = (\overline{X}-12)/(s/\sqrt{20}) = (11-12)/(4/\sqrt{20}) = -\sqrt{5}/2 = -1.118$. We reject H_0 if this is less than $-t_{0.99}(19) = -2.539$ (p. 603). It is not, so we do not reject H_0 at the 0.01 level.

(e) Test statistic is $V_0 = (n-1)S^2/9 = 19 \cdot 16/9 = 33.8$. We reject H_0 if $V_0 \geq \chi^2_{0.99}(19) = 36.19$. We do not reject H_0 at the 0.01 level.

(f) Use the result at the top of page 402. We want n such that $(9/18)\chi^2_{0.99}(n-1) = \chi^2_{0.10}(n-1)$. In other words, for what n is $\chi^2_{0.99}(n-1)$ twice as large as $\chi^2_{0.10}(n-1)$? By page 605, it looks like $n \approx 50$ suffices. If we use the approximation on page 402, we get n approximately equal to

$$1 + 2\left[\frac{z_{0.10} - (9/18)z_{0.99}}{1-(9/18)}\right]^2 = 1 + 2\left[\frac{-1.282 - (1/2)2.326}{1/2}\right]^2 = 48.8,$$

or $n = 49$. β is 1 minus the power, so $\beta \approx 0.90$.

5. Here $H_0 : \mu = 200$ vs. $H_a : \mu \neq 200$. Test statistic is $T_0 = (\overline{X} - 200)/(S/\sqrt{n})$. Critical region is $|T_0| \geq t_{1-\alpha/2}(n-1)$ with $\alpha = 0.01$. We use page 400 and Table 8 on page 612, in which $\alpha = 0.005$ because this is a two-sided test. Here $d = |\mu_1 - \mu_2|/\sigma = 20/25 = 0.8$. The table gives $n = 32$.