Math 5080
Solutions to Exam 1
10/11/2013

1. Notice that $X_2 = (X_1 + X_2)Y_2 = Y_1^2 Y_2$ and $X_1 = Y_1^2 - X_2 = Y_1^2 - Y_1^2 Y_2 = Y_1^2(1 - Y_2)$, so the inverse transformation is

$$x_1 = y_1^2(1 - y_2)$$
$$x_2 = y_1^2 y_2.$$

Its Jacobian is

$$J = \det \begin{pmatrix} 2y_1(1-y_2) & -y_1^2 \\ 2y_1 y_2 & y_1^2 \end{pmatrix} = 2y_1^3(1-y_2) + 2y_1^3 y_2 = 2y_1^3.$$

Now X_1 has pdf $f(x) = F'(x) = xe^{-x^2/2}$ for $x > 0$, so

$$\begin{aligned} g(y_1, y_2) &= f(x_1)f(x_2)|2y_1^3| = f(y_1^2(1-y_2))f(y_1^2 y_2)2y_1^3 \\ &= y_1^2(1-y_2)e^{-[y_1^2(1-y_2)]^2/2} y_1^2 y_2 e^{-[y_1^2 y_2]^2/2} 2y_1^3, \end{aligned}$$

and this holds for $y_1 > 0$ and $0 < y_2 < 1$ (which is equivalent to $x_1 > 0$ and $x_2 > 0$).

2. The equation

$$M_{X_1+X_2+X_3}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t)$$

becomes

$$e^{4t+(1/2)16t^2} = e^{2t+(1/2)4t^2} e^{3t+(1/2)9t^2} M_{X_3}(t),$$

therefore

$$M_{X_3}(t) = e^{(4-2-3)t+(1/2)(16-4-9)t^2} = e^{-t+(1/2)3t^2}.$$

The latter MGF is that of $N(-1, 3)$, so X_3 must be $N(-1, 3)$.

3. (a) The theorem tells us that $X_{\lfloor n/2 \rfloor:n}$ is asymptotically normal with asymptotic mean $x_{1/2}$ and asymptotic variance c^2/n, where $c^2 = \frac{1}{2}(1 - \frac{1}{2})/f(x_{1/2})^2$. Here $f(x) = e^{-x}$ for $x > 0$, and the corresponding CDF is $F(x) = 1 - e^{-x}$ for $x > 0$. Now $x_{1/2}$ satisfies $F(x_{1/2}) = \frac{1}{2}$, i.e.,

$$1 - e^{-x_{1/2}} = \frac{1}{2},$$

so $x_{1/2} = \ln 2$. Finally, $c^2 = 1/(2e^{-x_{1/2}})^2 = 1/(2(1/2))^2 = 1$. In summary, $X_{\lfloor n/2 \rfloor:n}$ is asymptotically normal with asymptotic mean $\ln 2$ and asymptotic variance $1/n$.

(b) By the theorem, Y_n^3 is asymptotically normal with asymptotic mean m^3 and asymptotic variance $c^2(3m^2)^2/n$, since y^3 has derivative $3y^2$.

4. (a) The simplest answer is $Y_1 = (X_1 - \mu)/\sigma$. Another is $Y_1' = (\overline{X}_4 - \mu)/(\sigma/\sqrt{4})$.
(b) The simplest answer is

$$Y_2 = \left(\frac{X_2 - \mu}{\sigma}\right)^2 + \left(\frac{X_3 - \mu}{\sigma}\right)^2 + \left(\frac{X_4 - \mu}{\sigma}\right)^2,$$

but $Y_2' = (4-1)S_4^2/\sigma^2$ also works.

(c) Take $Y_3 = Y_1^2/(Y_2/3)$. The only thing one must be careful of is to make sure that Y_1 and Y_2 are independent.

(d) Take $Y_4 = Y_1/\sqrt{Y_2/3}$. The only thing one must be careful of is to make sure that Y_1 and Y_2 are independent. Another possibility is $Y_4' = (\overline{X}_4 - \mu)/(S_4/\sqrt{4})$.