Math 5080-2 Name: ____________________________
Midterm Exam 2
Nov. 18, 2015

Directions: Solve 4 of the 5 problems. Do not attempt all 5. You may use one sheet of notes, but exchanging notes is not allowed.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the population with CDF (**cumulative** distribution function)

$$F(x; \alpha, \beta) = x^\alpha/\beta^\alpha, \qquad 0 \le x \le \beta$$

where α and β are positive. Find the corresponding density and then the maximum likelihood estimators of α and β.

Sol. Density is $f(x; \alpha, \beta) = \alpha\beta^{-\alpha}x^{\alpha-1}I_{[0,\beta]}(x)$. Likelihood function is

$$L(\alpha, \beta) = \prod_{i=1}^n f(x_i; \alpha, \beta) = \alpha^n\beta^{-n\alpha}(x_1x_2\cdots x_n)^{\alpha-1}I_{[0,\beta]}(x_{n:n}).$$

It follows by inspection that $\hat\beta = x_{n:n}$. It remains to determine $\hat\alpha$.

$$\ln L(\alpha, \hat\beta) = n\ln\alpha - n\alpha\ln\hat\beta + (\alpha - 1)\sum_i \ln x_i,$$

so

$$0 = \frac{\partial}{\partial\alpha}\ln L(\alpha, \hat\beta) = \frac{n}{\alpha} - n\ln\hat\beta + \sum_i \ln x_i,$$

hence

$$\hat\alpha = \frac{1}{\ln x_{n:n} - (1/n)\sum\ln x_i} = \frac{n}{\sum\ln(x_{n:n}/x_i)}.$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \theta)$, where $\theta > 0$.

(a) Find the Cramér–Rao lower bound for the variance of unbiased estimators of θ.

(b) Use this information to find a UMVUE of θ.

Hint: You may use the fact that $\mathrm{Var}_\theta(X_1^2) = 2\theta^2$. Incidentally, the $N(\mu, \sigma^2)$ density has the form

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty.$$

Sol. (a) $f(x; \theta) = (2\pi\theta)^{-1/2}e^{-x^2/(2\theta)}$, so

$$\ln f(x; \theta) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{x^2}{2\theta}, \qquad \frac{\partial}{\partial\theta}\ln f(x; \theta) = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2},$$

hence $I_1(\theta) = E_\theta[(X^2-\theta)^2]/(2\theta^2)^2 = \mathrm{Var}_\theta(X^2)/(4\theta^4) = 2\theta^2/(4\theta^4) = 1/(2\theta^2)$. Therefore, CRLB $= 1/(nI_1(\theta)) = 2\theta^2/n$.

Alternatively,

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

hence $I_1(\theta) = -E_\theta[\theta - 2X_1^2]/(2\theta^3) = \theta/(2\theta^3) = 1/(2\theta^2)$, etc.

(b) Try $T = (1/n)\sum_1^n X_i^2$, the method-of-moments estimator. Then $E_\theta[T] = E[X_1^2] = \theta$ and $\mathrm{Var}_\theta(T) = (1/n)\mathrm{Var}_\theta(X_1^2) = 2\theta^2/n$. Since T is unbiased and the CRLB is achieved, T is a UMVUE.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from BIN$(2,\theta)$, where $0<\theta<1$, i.e., X_i is $0,1,2$ with probabilities $(1-\theta)^2, 2\theta(1-\theta), \theta^2$, resp. More concisely, $P(X_i = x) = \binom{2}{x}\theta^x(1-\theta)^{2-x}$ for $x = 0,1,2$.

(a) If Θ has prior density $f(\theta) = 6\theta(1-\theta)$, $0<\theta<1$, find the posterior density of Θ given the sample.

(b) Find the Bayes estimator, assuming squared error loss.

Hint: The BETA(α,β) density, as a function of θ, has the form

$$f(\theta;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0<\theta<1,$$

and its mean is $\alpha/(\alpha+\beta)$. For example, the prior density is BETA[2,2].

Sol. The binomial density is $f(x;\theta) = \binom{2}{x}\theta^x(1-\theta)^{2-x}$, so the posterior density is proportional (in θ) to $\theta(1-\theta)\cdot\prod_1^n \theta^{x_i}(1-\theta)^{2-x_i}$, or

$$\theta^{1+\sum_1^n x_i}(1-\theta)^{1+2n-\sum_1^n x_i}.$$

This is the beta$(2+\sum_1^n x_i, 2+2n-\sum_1^n x_i)$ density.

(b) The mean of the posterior beta distribution is $(2+\sum_1^n x_i)/(4+2n)$, which can be written as

$$\frac{2+\sum_1^n x_i}{2(2+n)} = \frac{1}{2}\,\frac{2+n\overline{x}}{2+n} = \frac{2}{2+n}\,\frac{1}{2} + \frac{n}{2+n}\,\frac{\overline{x}}{2}.$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from UNIF$(-\theta,\theta)$, where $\theta>0$.

(a) Find a pair of jointly sufficient statistics.

(b) Find a single sufficient statistic based on the two jointly sufficient ones.

Sol. (a) $f(\boldsymbol{x};\theta) = \prod_1^n (2\theta)^{-1} I_{(-\theta,\theta)}(x_i) = (2\theta)^{-n} I_{(-\infty,\theta)}(x_{n:n}) I_{(-\theta,\infty)}(x_{1:n})$. By the factorization theorem, $x_{n:n}$ and $x_{1:n}$ are jointly sufficient.

(b) The last expression can be rewritten as $(2\theta)^{-n} I_{(-\infty,\theta)}(x_{n:n}) I_{(-\theta,\infty)}(x_{1:n}) = (2\theta)^{-n} I_{(-\infty,\theta)}(x_{n:n}) I_{(-\infty,\theta)}(-x_{1:n}) = (2\theta)^{-n} I_{(-\infty,\theta)}(\max\{x_{n:n}, -x_{1:n}\})$. By the factorization theorem, $\max\{x_{n:n}, -x_{1:n}\}$ is sufficient.

5. Return to the assumptions of Problem 2.

(a) Recalling the regular exponential class (REC), find a complete sufficient statistic.

(b) State the Lehmann–Scheffé theorem, and use it to find a UMVUE for θ.

Sol. (a) $f(x;\theta) = (2\pi\theta)^{-1}\exp(-(2\theta)^{-1}x^2)I_{(-\infty,\infty)}(x)$, which is REC with $S = \sum X_i^2$ complete and sufficient.

(b) LS Theorem: If S is complete and sufficient and $T = h(S)$ is unbiased for $\tau(\theta)$, then T is a UMVUE of $\tau(\theta)$.

Apply this with $\tau(\theta) = \theta$ and $T = S/n$ to get the desired result.