Math 5080-2 Name: solutions
Midterm Exam 1
October 7, 2015

Turn in solutions for any 3 of the 4 problems. If you attempt all 4 problems, each will count 1/4 instead of 1/3. You may use one formula sheet, but exchanging formula sheets is not allowed.

1. Let X_1 and X_2 be independent EXP(1) random variables (pdf e^{-x}, $x > 0$). Find the joint pdf of (a) $Y_1 = X_1/X_2$ and (b) $Y_2 = X_1 + X_2$.

Hint for inverse transformation: Solve eq. (a) for X_1 in terms of Y_1 and X_2 and call this eq. (c); solve eq. (b) for X_2 in terms of Y_2 and X_1 and call this eq. (d). Substitute eq. (d) into eq. (c) and solve for X_1.

Sol. (c) $X_1 = Y_1 X_2$. (d) $X_2 = Y_2 - X_1$. Hence $X_1 = Y_1(Y_2 - X_1) = Y_1 Y_2 - Y_1 X_1$ or $(1 + Y_1)X_1 = Y_1 Y_2$, hence

$$X_1 = \frac{Y_1 Y_2}{1 + Y_1}, \qquad X_2 = \frac{Y_2}{1 + Y_1}.$$

Jacobian is

$$J = \det \begin{pmatrix} y_2/(1+y_1)^2 & y_1/(1+y_1) \\ -y_2/(1+y_1)^2 & 1/(1+y_1) \end{pmatrix} = \frac{y_2 + y_1 y_2}{(1+y_1)^3} = \frac{y_2}{(1+y_1)^2},$$

Conclude that

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1}\left(\frac{y_1 y_2}{1+y_1}\right) f_{X_2}\left(\frac{y_2}{1+y_1}\right) \frac{y_2}{(1+y_1)^2} = y_2 e^{-y_2}/(1+y_1)^2,$$

where $y_1 > 0$ and $y_2 > 0$.

2. Let X_1, X_2, X_3 be a random sample of size 3 from the UNIF(0, 1) distribution.

(a) Find the joint pdf of the order statistics Y_1, Y_2, Y_3. Part of the problem is to identify the y_1, y_2, y_3 values for which the formula is valid.

(b) Find the (marginal) joint pdf of Y_1 and Y_3. Part of the problem is to identify the y_1, y_3 values for which the formula is valid.

(c) Find the mean of the sample range $R = Y_3 - Y_1$. (It is not necessary to find the pdf of R.)

Sol. (a) By a theorem, $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = 6$, $0 < y_1 < y_2 < y_3 < 1$.

(b) $f_{Y_1,Y_3}(y_1, y_3) = \int_{y_1}^{y_3} f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)\, dy_2 = 6(y_3 - y_1)$, $0 < y_1 < y_3 < 1$.

(c) $E[Y_3 - Y_1] = \int\int (y_3 - y_1) 6(y_3 - y_1)\, dy_1\, dy_3 = \int_0^1 \int_{y_1}^1 6(y_3 - y_1)^2\, dy_3\, dy_1 = \int_0^1 2(1 - y_1)^3\, dy_1 = \frac{1}{2}$.

Alternatively, $E[Y_3 - Y_1] = E[Y_3] - E[Y_1] = \int_0^1 y_3 3y_3^2\, dy_3 = \int_0^1 y_1 3(1 - y_1)^2\, dy_1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

3. Let $X_1, X_2, \ldots, X_n$ be independent LOGN(μ, σ^2) (i.e., $\ln(X_1), \ln(X_2), \ldots, \ln(X_n)$ are independent N(μ, σ^2)). Use a theorem to show that the sample

median is asymptotically normal and find its asymptotic mean and asymptotic variance. For simplicity, assume that n is odd. In case the LOGN density is needed, it has the form $f(x) = (2\pi\sigma^2)^{-1/2}x^{-1}e^{-(\ln x-\mu)^2/(2\sigma^2)}$, $x > 0$.

Sol. $X_1 = e^{Y_1}, \ldots, X_n = e^{Y_n}$ with $Y_1, \ldots, Y_n$ i.i.d. N(μ, σ^2). What is $x_{1/2}$ for X_1? $\frac{1}{2} = P(X_1 > x_{1/2}) = P(Y_1 > \ln x_{1/2})$ implies $\ln x_{1/2} = \mu$ or $x_{1/2} = e^\mu$. Conclude by a theorem that the sample median $X_{(n+1)/2:n}$ is asymptotically normal with asymptotic mean $x_{1/2} = e^\mu$ and asymptotic variance c^2/n, where

$$c^2 = \frac{\frac{1}{2}(1-\frac{1}{2})}{f_{X_1}(x_{1/2})^2} = \frac{1/4}{[(\sqrt{2\pi\sigma^2}\, e^\mu)^{-1}]^2} = (\pi/2)\sigma^2 e^{2\mu}.$$

Alternative solution: $Y_{(n+1)/2:n}$ is asymptotically normal with asymptotic mean μ and asymptotic variance c^2/n, where

$$c^2 = \frac{\frac{1}{2}(1-\frac{1}{2})}{f_{Y_1}(\mu)^2} = \frac{1/4}{[(\sqrt{2\pi\sigma^2})^{-1}]^2} = (\pi/2)\sigma^2.$$

Now $X_{(n+1)/2:n} = e^{Y_{(n+1)/2:n}}$, which is therefore asymptotically normal with asymptotic mean e^μ and asymptotic variance $e^{2\mu}c^2/n$ with c^2 as above.

4. Let V_n be χ^2 with n degrees of freedom.

(a) Show that $(1/n)V_n$ is asymptotically normal, and determine the asymptotic mean and asymptotic variance. Hint: If Z is $N(0,1)$, then $\mathrm{Var}(Z^2) = 2$.

(b) Explain why $e^{(1/n)V_n}$ is asymptotically normal, and find its asymptotic mean and asymptotic variance.

Sol. (a) By the central limit theorem, $\overline{X}_n$ is asymptotically normal with asymptotic mean μ and asymptotic variance σ^2/n. Now $(1/n)V_n$ is of the form of $\overline{X}_n$ with $\mu = E[Z^2] = 1$ and $\sigma^2 = \mathrm{Var}(Z^2) = 2$, so $(1/n)V_n$ is asymptotically normal with asymptotic mean 1 and asymptotic variance $2/n$.

(b) We apply a theorem with $g(x) = e^x$, hence $g'(x) = e^x$ also. We get that $e^{(1/n)V_n}$ is asymptotically normal with asymptotic mean $g(1) = e$ and asymptotic variance $g'(1)^2(2/n) = 2e^2/n$.