1. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Compute the density function of (Y_1, Y_2), where $Y_1 = X_1$ and $Y_2 = X_1 X_2$.

$x_1 = y_1$

$x_2 = y_2/y_1$

$$J = \begin{bmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{bmatrix}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1, y_2/y_1)\frac{1}{|y_1|} \cdot 1\{y_1 > 0,\ y_2 > 0\}$$

$$= \begin{cases} e^{-y_1 - y_2/y_1} \dfrac{1}{y_1} & y_1 > 0,\ y_2 > 0, \\ 0 & \text{o/w.} \end{cases}$$

2. Let N be a standard normal random variable. Compute the moment generating function of $|N|$. (The moment generating function can be written with the help of the standard normal distribution function Φ.)

$$E\left(e^{t|N|}\right) = \int_{\mathbb{R}} e^{t|x|} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + t|x|} dx$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + tx} dx$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} e^{t^2/2} dx$$

$$= 2e^{t^2/2}\int_{-t}^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 2e^{t^2/2}\left[\Phi(\infty) - \Phi(-t)\right]$$

$$= 2e^{t^2/2}\left[1 - \Phi(-t)\right]$$

$$= 2e^{t^2/2}\,\Phi(t)$$

3. Let X_1 and X_2 be independent random variables with density functions

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}e^{-(x-1)/2} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases}$$

respectively. Compute the density function of $X_1 + X_2$.

$$f_{X_1+X_2}(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$$

$$= \int_{\mathbb{R}} e^{-y}\, 1\{y>0\}\, \frac{1}{2} e^{-(x-y-1)/2}\, 1\{x-y>1\}\, dy$$

$$= \begin{cases} \int_0^{x-1} e^{-y} \frac{1}{2} e^{-\frac{(x-1)}{2}} e^{y/2}\, dy & x > 1, \\ 0 & o/w. \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{\frac{1-x}{2}} \int_0^{x-1} e^{-y/2}\, dy & x > 1, \\ 0 & o/w. \end{cases}$$

$$= \begin{cases} e^{\frac{1-x}{2}} \left(1 - e^{\frac{1-x}{2}}\right) & x > 1, \\ 0 & o/w. \end{cases}$$

4. Let X_1, X_2, ..., X_n be independent identically distributed random variables with cumulative distribution function

$$F(x) = \begin{cases} 1 - 1/x & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Find the limiting distribution of $X_{1:n}^n$.

$$F_{X_{1:n}^n}(y) = \mathbb{P}(X_{1:n}^n \leq y) = \begin{cases} \mathbb{P}(X_{1:n} \leq y^{1/n}) & y > 1, \\ 0 & y \leq 1. \end{cases}$$

$$= \begin{cases} 1 - \mathbb{P}(X_{1:n} > y^{1/n}) & y > 1 \\ 0 & y \leq 1 \end{cases} = \begin{cases} 1 - [1 - F(y^{1/n})]^n & y > 1 \\ 0 & y \leq 1 \end{cases}$$

$$= \begin{cases} 1 - [1 - (1 - \frac{1}{y^{1/n}})]^n & y > 1 \\ 0 & y \leq 1 \end{cases} = \begin{cases} 1 - [\frac{1}{y^{1/n}}]^n & y > 1 \\ 0 & y \leq 1 \end{cases}$$

$$= \begin{cases} 1 - \frac{1}{y} & y > 1 \\ 0 & y \leq 1 \end{cases}$$

5. Let X_1, X_2, ..., X_n be independent identically distributed random variables with density function

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Approximate $P\left(\sum_{i=1}^{90} X_i \leq 77\right)$ in terms of $\Phi(\cdot)$, the cdf of a standard normal.

$$\mathbb{P}\left(\frac{\sum X_i - 45}{\sqrt{\frac{90}{12}}} \leq \frac{77-45}{\sqrt{\frac{90}{12}}}\right) \approx \Phi\left(\frac{32}{\sqrt{\frac{15}{2}}}\right)$$

Extra Credit on Exam 1

1) $1 - e^{-X}$.

2) $1 - 2e^{-X}$.

3) Yes.

4) No. Let $Y_n = -X$ where $X \sim N(0,1)$.

Then $Y_n \sim N(0,1)$, so $Y_n \xrightarrow{d} X$.

However, $\mathbb{P}\left(|Y_n - X| > \varepsilon\right) = \mathbb{P}(2|X| > \varepsilon) = g(\varepsilon) > 0$.

Since $g(\varepsilon)$ does not depend on n, it will not converge to zero.

5) If $X_i \sim iid\,(\mu, \sigma^2)$ with μ, σ^2 finite,
then $\bar{X}_n \xrightarrow{\mathbb{P}} \mu$.

6) If $X_i \sim iid\,(\mu, \sigma^2)$ with μ, σ^2 finitie,
then
$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1),$$

i.e. $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$.