should be $\frac{e^x}{e-1}$

1. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} \frac{e^t}{e-1} & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Compute the moment generating function of $2X_1 - 3X_2 + 2$.

$$M_X(t) = E(e^{tX}) = \int_0^1 \frac{e^{tx} e^{x}}{e-1}\, dx = \int_0^1 \frac{e^{(t+1)x}}{e-1}\, dx$$

$$= \begin{cases} \left.\frac{e^{(t+1)x}}{(t+1)(e-1)}\right|_0^1 = \frac{e^{t+1} - 1}{(t+1)(e-1)} & t \neq -1 \\ \frac{1}{e-1} & t = -1 \end{cases}$$

$$M_{2X_1 - 3X_2 + 2}(t) = M_X(2t)\, M_X(-3t)\, e^{2t}$$

$$= \begin{cases} \left[\frac{e^{2t+1} - 1}{(2t+1)(e-1)}\right]\left[\frac{e^{-3t+1} - 1}{(-3t+1)(e-1)}\right] e^{2t} & t \notin \{-\frac{1}{2}, \frac{-1}{3}\} \\ \left[\frac{1}{e-1}\right]\left[\frac{e^{-3t+1} - 1}{(-3t+1)(e-1)}\right] e^{2t} & t = -\frac{1}{2} \\ \left[\frac{e^{2t+1} - 1}{(2t+1)(e-1)}\right]\left[\frac{1}{e-1}\right] e^{2t} & t = \frac{-1}{3} \end{cases}$$