1. Consider a random sample of size n from a distribution with pdf $f(x;\theta) = \frac{1}{\theta}$ if $0 < x \leq \theta$, and zero otherwise; $0 < \theta$. Compute the MLE and MME for $\theta$. Compare the bias and MSE of the two estimators.

$L(\theta) = \frac{1\{\theta > X_{nn}\}}{\theta^n}$. $\boxed{X_{n:n} \text{ is the MLE for } \theta.}$

$\text{Bias}(X_{n:n}) = E(X_{n:n} - \theta)$.

$F_{X_{n:n}}(y) = P(X_{n:n} \leq y) = [F_X(y)]^n = \left(\frac{y}{\theta}\right)^n$ when $y \in (0,\theta]$.

$f_{X_{n:n}}(y) = \frac{ny^{n-1}}{\theta^n} 1\{y \in (0,\theta]\}$.

$E(X_{n:n}) = \int_0^\theta \frac{y\,n\,y^{n-1}}{\theta^n}\,dy = \left(\frac{n}{\theta^n}\right)\frac{y^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{\theta^n(n+1)}\left[\theta^{n+1}\right] = \frac{n\theta}{n+1}$.

$\boxed{\text{So, bias}(X_{n:n}) = \theta\left[\frac{n}{n+1} - 1\right] = \frac{-\theta}{n+1}}$

$MSE(X_{n:n}) = E(X_{n:n} - \theta)^2 = E(X_{n:n}^2) - 2\theta E(X_{n:n}) + \theta^2$

$E(X_{n:n}^2) = \int_0^\theta \frac{ny^{n+1}}{\theta^n}\,dy = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$

So, $MSE(X_{n:n}) = \frac{n\theta^2}{n+2} - 2\theta\left(\frac{n\theta}{n+1}\right) + \theta^2 = \frac{(n+1)n\theta^2 - 2n(n+2)\theta^2 + (n+1)(n+2)\theta^2}{(n+1)(n+2)}$

$= \frac{\theta^2[n^2+n-2n^2-4n+n^2+2+3n]}{(n+1)(n+2)} = \boxed{\frac{2\theta^2}{(n+1)(n+2)}}$

For MME: $\frac{\theta}{2} = \bar{X} \Rightarrow \boxed{\hat{\theta} = 2\bar{X} \text{ is the MME.}}$

$\boxed{\text{Bias}(2\bar{X}) = E(2\bar{X} - \theta) = 0.}$

$MSE(2\bar{X}) = var(2\bar{X}) + [\text{bias}(2\bar{X})]^2 = var(2\bar{X}) = 4\,var(\bar{X}) = \frac{4\,var(X)}{n}$.

$var(X) = E(X^2) - \left(\frac{\theta}{2}\right)^2$. $E(X^2) = \int_0^\theta x^2/\theta\,dx = \frac{x^3}{3\theta}\Big|_0^\theta = \frac{\theta^2}{3}$.

Thus $var(X) = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$ and $\boxed{MSE(2\bar{X}) = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}}$.

So, the MME is unbiased but has larger MSE than the biased MLE for reasonable n.