Assume σ_1^2 and σ_2^2 are known.

1. Consider independent random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ from normal distributions with a common mean, μ, but with possibly different variances, σ_1^2 and σ_2^2, so that $X_i \sim N(\mu, \sigma_1^2)$ and $Y_i \sim N(\mu, \sigma_2^2)$. Find the MLEs of μ, ~~σ_1^2, σ_2^2~~.

$$L(\mu, \sigma_1^2, \sigma_2^2) = \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-(x_i-\mu)^2}{2\sigma_1^2}} \right) \left(\prod_{k=1}^{m} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{\frac{-(y_k-\mu)^2}{2\sigma_2^2}} \right)$$

let $\theta_1 = \sigma_1^2$; $\theta_2 = \sigma_2^2$.

$$\ell(\mu, \theta_1, \theta_2) = \sum_{i=1}^{n} \left(C - \frac{1}{2}\log(\theta_1) - \frac{(x_i-\mu)^2}{2\theta_1} \right)$$

$$+ \sum_{k=1}^{m} \left(D - \frac{1}{2}\log(\theta_2) - \frac{(y_k-\mu)^2}{2\theta_2} \right)$$

$$= nC - \frac{n}{2}\log\theta_1 + mD - \frac{m}{2}\log(\theta_2)$$

$$- \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{2\theta_1} - \frac{\sum_{k=1}^{m}(y_k-\mu)^2}{2\theta_2}$$

$$\frac{\partial \ell}{\partial \mu} = + \frac{\cancel{2}\sum_{i=1}^{n}(X_i-\mu)}{\cancel{2}\theta_1} + \frac{\sum_{k=1}^{m}(y_k-\mu)}{\theta_2} = 0$$

$$\Rightarrow \theta_2 \sum_{i=1}^{n}(X_i-\mu) + \theta_1 \sum_{k=1}^{m}(y_k-\mu) = 0$$

$$\Rightarrow \theta_2 \sum_{i=1}^{n} X_i - n\theta_2\mu + \theta_1 \sum_{k=1}^{m} y_k - m\theta_1\mu = 0$$

$$\Rightarrow \theta_2 \sum_{i=1}^{n} X_i + \theta_1 \sum_{k=1}^{m} y_k = \mu(\theta_1 m + n\theta_2)$$

$$\Rightarrow \mu = \frac{\theta_2 \sum x_i + \theta_1 \sum y_k}{m\theta_1 + n\theta_2} = \left(\frac{n\theta_2}{m\theta_1 + n\theta_2}\right)\bar{x} + \left(\frac{m\theta_1}{m\theta_1 + n\theta_2}\right)\bar{y}$$