1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \frac{1}{2\theta}e^{-|t|/\theta}, \quad -\infty < t < \infty.$$

Find the maximum likelihood estimator for θ and compute the asymptotic variance of the maximum likelihood estimator.

$L(\theta) = \prod \frac{1}{2\theta} e^{-|x_i|/\theta} \mathbb{1}\{\theta > 0\}$. If $\theta > 0$, then

$\ell(\theta) = \sum\left(-\log(2) - \log(\theta) - \frac{|x_i|}{\theta}\right) = C - n\log(\theta) - \frac{1}{\theta}\sum|x_i|.$

$$\frac{d\ell}{d\theta} = \frac{-n}{\theta} + \frac{\sum|x_i|}{\theta^2} = 0$$

$$\Rightarrow \quad -n\theta + \sum|x_i| = 0$$

$$\Rightarrow \quad \theta = \frac{\sum|x_i|}{n}$$

Thus $\hat{\theta} = \frac{\sum_{i=1}^{n}|x_i|}{n}$.

$$\text{CRLB} = \frac{1}{nE\left[\frac{\partial}{\partial\theta}\log f(X;\theta)\right]^2} = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]}$$

$$\log f(X;\theta) = \frac{-|X|}{\theta} - \log(2) - \log(\theta)$$

$$\frac{\partial}{\partial\theta}\log f(X;\theta) = \frac{+|X|}{\theta^2} - \frac{1}{\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) = \frac{-2|X|}{\theta^3} + \frac{1}{\theta^2}$$

$$E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] = \frac{-2}{\theta^3}E|X| + \frac{1}{\theta^2}$$

$$E(|X|) = \int \frac{|x|}{2\theta} e^{-|x|/\theta}\,dx = 2\int_0^\infty \frac{1}{2\theta} x e^{-x/\theta} = \theta$$

Asymptotic variance of $\hat{\theta}$ is

$$\frac{1}{-n\left(-\frac{2}{\theta^2} + \frac{1}{\theta^2}\right)} = \frac{1}{n\left(\frac{1}{\theta^2}\right)}$$

$$= \frac{\theta^2}{n}$$