1. Let $N_1, \ldots, N_5$ be independent normal random vaiables with zero mean. Asuume $EN_1^2 = EN_2^2 = EN_3^2 = 4$ and $EN_4^2 = EN_5^2 = 9$. Find c such that

$$.95 = P\left\{N_1^2 + N_2^2 + N_3^2 \le c(N_4^2 + N_5^2)\right\}.$$

$$.95 = \mathbb{P}\left(\frac{N_1^2 + N_2^2 + N_3^2}{4} \le \frac{9c}{4}\left(\frac{N_4^2 + N_5^2}{9}\right)\right)$$

$$= \mathbb{P}\left[\frac{\left(\frac{N_1^2 + N_2^2 + N_3^2}{4}\right)/3}{\left(\frac{N_4^2 + N_5^2}{9}\right)/2} \le \frac{(2)9c}{4(3)}\right]$$

$$= \mathbb{P}\left(Y \le \frac{3c}{2}\right) \quad \text{where} \quad Y \sim F(3,2)$$

$$\Rightarrow \frac{3c}{2} = f_{.95}(3,2) = 19.164$$

$$\Rightarrow c = \frac{2}{3}(19.164)$$

2. Let X_1, X_2 and X_3 be three independent normal random variables. X_1 is normal $N(0,4)$, X_2 is normal $N(0,12)$ and X_3 is normal $N(0,9)$. Compute

$$P\left\{\frac{X_1+X_2}{|X_3|} \leq 3\right\}$$

using one of the enclosed tables.

$X_1+X_2 \sim N(0,4^2)$.

$$\mathbb{P}\left(\frac{X_1+X_2}{|X_3|} \leq 3\right) = \mathbb{P}\left[\frac{\left(\frac{X_1+X_2}{4}\right)}{\sqrt{\frac{X_3^2}{9}}} \leq \frac{3}{4}\cdot 3\right]$$

$$= \mathbb{P}\left(Y \leq \frac{9}{4}\right) \quad \text{where} \quad Y \sim t(1)$$

$$= .87$$

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t) = \begin{cases} 0, & \text{if } t < \theta \\ e^{-(t-\theta)}, & \text{if } t > \theta. \end{cases}$$

Find a moment estimator for θ, where θ is the unknown parameter.

$\bar{X} = E(X) = \theta + 1 \quad \Rightarrow \quad \theta = \bar{X} - 1$

So a MME for θ is $\hat{\theta} = \bar{X} - 1$.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } t \notin [1,3] \\ \theta 2^{-\theta}(t-1)^{\theta-1}, & \text{if } t \in [1,3] \end{cases}$$

Find a maximum liklihood estimator for θ.

$$L(\theta) = \prod_{i=1}^{n} \theta 2^{-\theta}(x_i - 1)^{\theta-1} \, 1\{\theta > 0\}$$

$$= \theta^n 2^{-n\theta} \prod_{i=1}^{n} (x_i - 1)^{\theta - 1} \, 1\{\theta > 0\}.$$

Assume $\theta > 0$.

Then $\ell(\theta) = n\log(\theta) - n\theta\log(2) + (\theta - 1)\sum_{i=1}^{n} \log(x_i - 1)$.

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - n\log(2) + \sum_{i=1}^{n} \log(x_i - 1) = 0$$

$$\Rightarrow \quad \frac{n}{\theta} = n\log(2) - \sum \log(x_i - 1)$$

$$\Rightarrow \quad \theta = \frac{n}{n\log(2) - \sum \log(x_i - 1)}$$

Thus $\hat{\theta} = \dfrac{n}{n\log(2) - \sum \log(X_i - 1)}$ is the MLE for θ.

5. Let X be a random variable with density function

$$h(t;\theta) = \frac{1}{2\theta} e^{-|t|/\theta}, \quad -\infty < t < \infty.$$

Compute the Cramer-Rao lower bound. (for Θ)

$$\text{CRLB} = \frac{[\tau'(\Theta)]^2}{-n\, E\left(\frac{\partial^2}{\partial\Theta^2} \log h(X;\theta)\right)} = \frac{[\tau'\Theta]^2}{n\, E\left[\frac{\partial}{\partial\Theta} \log h(X;\theta)\right]^2}$$

$$\log h(X;\Theta) = -\log(2) - \log(\Theta) - \frac{|X|}{\Theta}$$

$$\frac{\partial}{\partial\theta} \log h(X;\theta) = \frac{-1}{\Theta} + \frac{|X|}{\Theta^2}$$

$$\frac{\partial^2}{\partial\Theta^2} \log h(X;\Theta) = \frac{1}{\Theta^2} - \frac{2|X|}{\Theta^3}$$

$$E|X| = \int_{\mathbb{R}} |x| \frac{1}{2\Theta} e^{-|x|/\Theta} dx = 2\int_0^\infty \frac{x}{2\Theta} e^{-x/\Theta} dx = \Theta$$

$$E\left[\frac{\partial^2}{\partial\Theta^2} \log h(X;\Theta)\right] = \frac{1}{\Theta^2} - \frac{2\Theta}{\Theta^3} = \frac{1}{\theta^2} - \frac{2}{\Theta^2} = \frac{-1}{\Theta^2}$$

CRLB for Θ is $\dfrac{1}{-n\left[\frac{-1}{\theta^2}\right]} = \boxed{\dfrac{\Theta^2}{n}.}$