Must Assume $\theta > 1$ so that $E(X)$ exists.

5080–Quiz

Name:

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } -\infty < t < 1 \\ \theta t^{-(\theta+1)}, & \text{if } 1 \le t < \infty \end{cases}$$

(a) Find a moment estimator for θ.

$$\bar{X} = E(X)\Big|_{\theta=\hat{\theta}} = \int_1^\infty \hat{\theta} x^{-\hat{\theta}} \, dx = \frac{\hat{\theta} x^{-\hat{\theta}+1}}{-\hat{\theta}+1}\Bigg|_{x=1}^{\infty} = \frac{-\hat{\theta}}{1-\hat{\theta}}$$

$$\Rightarrow (1-\hat{\theta})\bar{X} = -\hat{\theta}$$

$$\Rightarrow \bar{X} - \hat{\theta}\bar{X} + \hat{\theta} = 0$$

$$\Rightarrow \hat{\theta}(1-\bar{X}) = -\bar{X}$$

$$\Rightarrow \hat{\theta} = \frac{-\bar{X}}{1-\bar{X}}$$

(b) Find the maximum likelihood estimator for θ.

$$L(\theta) = \prod_{i=1}^{n} \theta X_i^{-(\theta+1)} \quad \underline{1}\{\theta > 0\}$$

Assume $\theta > 0$.

$$\ell(\theta) = \sum_{i=1}^{n} \left(\log(\theta) \quad -(\theta+1) \log(X_i) \right) = n \log(\theta) - (\theta+1) \sum_{i=1}^{n} \log(X_i)$$

$$\left.\frac{d\ell}{d\theta}\right|_{\theta=\hat{\theta}} = \left.\frac{n}{\theta} - \sum \log(X_i)\right|_{\theta=\hat{\theta}} = 0$$

$$\Rightarrow \quad \frac{n}{\hat{\theta}} = \sum \log(X_i)$$

$$\Rightarrow \quad \hat{\theta} = \frac{n}{\sum \log(X_i)}$$