5080–Quiz

There is a problem on the back.

Name:

1. Consider a random sample of size n from a distribution with CDF $F(x) = (1-\frac{1}{x})1\{x > 1\}$. Find the limiting distribution of $X^n_{1:n}$.

For $x > 1$,

$$F_{X^n_{1:n}}(x) = \mathbb{P}(X^n_{1:n} \leq x) = \mathbb{P}(X_{1:n} \leq x^{1/n})$$

$$= 1 - \mathbb{P}(X_{1:n} > x^{1/n}) = 1 - \left(1 - \left(1 - \frac{1}{x^{1/n}}\right)\right)^n$$

$$= 1 - \left(\frac{1}{x^{1/n}}\right)^n = 1 - \frac{1}{x}.$$

Thus
$$F_{X^n_{1:n}}(x) = \begin{cases} 1 - \frac{1}{x} & x > 1, \\ 0 & x \leq 1. \end{cases}$$

$$F_X(x) = \begin{cases} x^2 & x \in (0,1), \\ 0 & x \le 0, \\ 1 & x \ge 1. \end{cases}$$

2. Consider a random sample of size n from a distribution with density function, $f(x) = 2x \cdot 1\{x \in (0,1)\}$. Compute the limiting distribution of $n(1 - X_{n:n})$.

For $x \in (0, n)$,

$$F_{n(1-X_{n:n})}(x) = \mathbb{P}\left(n(1 - X_{n:n}) \le x\right) 1\{x \in (0,n)\}$$

$$= \mathbb{P}\left(1 - X_{n:n} \le \frac{x}{n}\right) 1\{x \in (0,n)\}$$

$$= \mathbb{P}\left(-X_{n:n} \le \frac{x}{n} - 1\right) 1\{x \in (0,n)\}$$

$$= \mathbb{P}\left(X_{n:n} > 1 - \frac{x}{n}\right) 1\{x \in (0,n)\}$$

$$= \left[1 - \mathbb{P}\left(X_{n:n} \le 1 - \frac{x}{n}\right)\right] 1\{x \in (0,n)\}$$

$$= \left[1 - \left(\left(1 - \frac{x}{n}\right)^2\right)^n\right] 1\{x \in (0,n)\}$$

$$= \left[1 - \left(1 - \frac{x}{n}\right)^{2n}\right] 1\{x \in (0,n)\}$$

$$= \left[1 - \left(1 - \frac{2x}{m}\right)^m\right] 1\{x \in (0, \tfrac{m}{2})\}$$

2

where $m = 2n$

$$\longrightarrow \left(1 - e^{-2x}\right) 1\{x > 0\}$$

Thus $n(1 - X_{n:n})$ is exponentially distributed with mean $\frac{1}{2}$.