5080–Quiz

There is a problem on the back.

Name:

Assume X_1 is also independent of X_2, X_3, X_4.

1. Let X_2, X_3, X_4 be independent random variables with mass function

$$f(x) = \frac{e^{-2}2^x}{x!}1\{x \in \{0, 1, 2, ...\}\}.$$

Suppose $Y = X_1 + X_2 + X_3 + X_4$ has mass function

$$f(x) = \frac{e^{-10}10^x}{x!}1\{x \in \{0, 1, 2, ...\}\}.$$

Find the mass function of $X_1 + X_3$.

The MGF of a $POI(\lambda)$ is $E\left(e^{tX}\right) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

$$M_Y(t) = e^{10(e^t - 1)} = M_{X_1}(t) \left[e^{2(e^t-1)}\right]^3$$

$$\Rightarrow M_{X_1}(t) = e^{4(e^t - 1)} \quad \text{so} \quad X_1 \sim POI(4)$$

$$M_{X_1+X_3}(t) = M_{X_1}(t) M_{X_3}(t) = e^{4(e^t-1)} e^{2(e^t-1)} = e^{6(e^t-1)}$$

So $X_1 + X_3 \sim POI(6)$ with mass function

$$p(x) = \begin{cases} \frac{e^{-6} 6^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & o/w. \end{cases}$$

$$y \le \frac{w-x}{2}$$

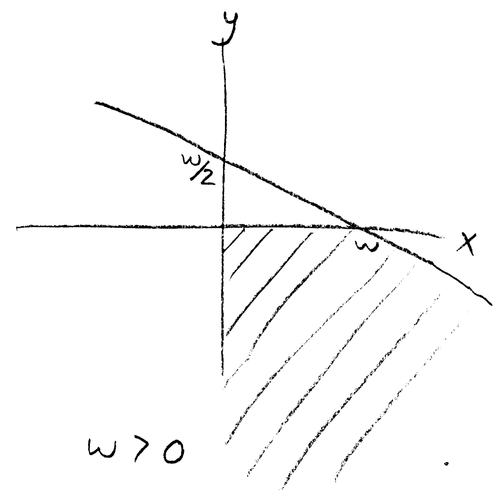
2. Let X and Y be independent random variables with density functions

$$g(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ e^{-t}, & \text{if } 0 \le t < \infty \end{cases}$$

and

$$h(t) = \begin{cases} e^{t}, & \text{if } t < 0 \\ 0, & \text{if } t \ge 0. \end{cases}$$

Compute the density of $X + 2Y$.



$$F_{X+2Y}(w) = \mathbb{P}(X+2Y \le w) = \begin{cases} \int_0^w \int_{-\infty}^0 e^{-x} e^{y}\, dy\, dx + \int_w^\infty \int_{-\infty}^{\frac{w-x}{2}} e^{-x} e^{y}\, dy\, dx & w > 0 \\ \int_0^\infty \int_{-\infty}^{\frac{w-x}{2}} e^{-x} e^{y}\, dy\, dx & w \le 0 \end{cases}$$

$$= \begin{cases} \int_0^w e^{-x} \cdot 1\, dx + \int_w^\infty e^{-x} e^{\frac{w-x}{2}}\, dx & w > 0, \\ \int_0^\infty e^{-x} e^{\frac{w-x}{2}}\, dx & w \le 0. \end{cases}$$

$$= \begin{cases} -e^{-x}\Big|_0^w + e^{w/2} e^{-\frac{3x}{2}} \left(\frac{-2}{3}\right)\Big|_w^\infty & w > 0, \\ e^{w/2} e^{-\frac{3x}{2}} \left(\frac{-2}{3}\right)\Big|_0^\infty & w \le 0. \end{cases} = \begin{cases} 1 - e^{-w} + \frac{2}{3} e^{-w} & w > 0, \\ \frac{2}{3} e^{w/2} & w \le 0. \end{cases}$$

Thus $f_{X+2Y}(w) = \begin{cases} \frac{1}{3} e^{-w} & w > 0, \\ \frac{1}{3} e^{w/2} & w \le 0. \end{cases}$

Other options: 1) convolution formula after transformation method

2) Joint transformation followed by marginal.