5080–Quiz

Name:

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } t \notin [0,1] \\ \theta t^{\theta-1}, & \text{if } t \in [0,1]. \end{cases}$$

(1) Find the maximum likelihood estimator for θ.

Assume $\theta > 0$.

$$L(\theta) = \prod \theta x_i^{\theta-1}$$

$$\ell(\theta) = \sum\left(\log(\theta) + (\theta-1)\log(x_i)\right)$$

$$= n\log(\theta) + (\theta-1)\sum \log(x_i)$$

$$\frac{d}{d\theta}\ell(\theta) = \frac{n}{\theta} + \sum \log(x_i) = 0$$

$$\Rightarrow \frac{n}{\theta} = -\sum \log(x_i)$$

$$\Rightarrow \theta = \frac{n}{-\sum \log(x_i)}$$

Thus $\hat{\theta} = \dfrac{n}{-\sum \log(X_i)}$.

(2) Compute the asymptotic variance of the maximum likelihood estimator.

The asymptotic variance of the unique MLE is the CRLB:

$$\frac{[\tau'(\theta)]^2}{n\, E\left[\frac{\partial}{\partial\theta} \log(f(X;\theta))\right]} \quad \text{i.e.} \quad \frac{[\tau'(\theta)]^2}{-n\, E\left[\frac{\partial^2}{\partial\theta^2} \log(f(X;\theta))\right]}$$

In this problem, $\tau'(\theta) = 1$

$$\cancel{\frac{\partial^2}{\partial\theta^2} = \partial}$$

$$\frac{\partial}{\partial\theta} \log f(X;\theta) = \frac{\partial}{\partial\theta}\left[\log(\theta) + (\theta-1)\log(X)\right]$$

$$= \frac{1}{\theta} + \log X$$

$$\frac{\partial^2}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{1}{\theta} + \log(X)\right) = \frac{-1}{{}_2\ \theta^2}$$

Thus the asymptotic var of $\hat{\theta}$ is $\dfrac{1}{-n\left(\frac{-1}{\theta^2}\right)} = \dfrac{\theta^2}{n}$.